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Boundary Effects in Potential Vorticity Dynamics

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ABSTRACT

Many aspects of geophysical flows can be described compactly in terms of potential vorticity dynamics. The fact, however, that the potential temperature can fluctuate at boundaries, and the implied inhomogeneous boundary condition, complicates considerations of potential vorticity dynamics of flows for which boundary effects are dynamically significant.

It is shown that the inhomogeneous boundary condition for potential vorticity dynamics can be replaced by a homogeneous boundary condition of constant potential temperature if the simplification of the boundary condition is compensated by a generalization of the potential vorticity concept to a sum of the conventional interior potential vorticity and a singular surface potential vorticity. Functional forms of the surface potential vorticity are derived from field equations in which the potential vorticity and a potential vorticity flux appear as sources of flow fields in the same way in which an electric charge and an electric current appear as sources of fields in electrodynamics. For the generalized potential vorticity of flows that need be neither balanced nor hydrostatic and that can be influenced by diabatic processes and friction, a conservation law holds that is similar to the conservation law for the conventional interior potential vorticity. The conservation law for generalized potential vorticity contains, in the quasigeostrophic limit, the well-known dual relationship between fluctuations of potential temperature at a boundary and fluctuations of potential vorticity in the interior of quasigeostrophic flows. A non-geostrophic effect described by the conservation law is the induction of generalized potential vorticity by baroclinicity at a boundary, an effect that plays a role, for example, in mesoscale flows past topographic obstacles. Based on the generalized potential vorticity concept, a theory is outlined of how a wake with lee vortices can form in weakly dissipative flows past a mountain that has no adjacent frictional boundary layer.

Replacing the inhomogeneous boundary condition for potential vorticity dynamics by a homogeneous boundary condition of constant potential temperature means that the flow domain in isentropic coordinates becomes time-independent, which allows one to consider a mean budget of generalized potential vorticity throughout the entire entropic flow domain, including the surface layer of isentropes that sometimes intersect the surface. From the mean budget of generalized potential vorticity, a balance condition is deduced that relates the extratropical mean meridional mass flux along isentropes to eddy fluxes of interior potential vorticity and of surface potential temperature. The generalized potential vorticity concept can therefore form a basis of theories of the mean meridional mass flux along isentropes and of the thermal stratification of the extratropical atmosphere.

1. Introduction

Since the potential vorticity is materially conserved in adiabatic and frictionless flows and since it contains all relevant information about balanced flows in a single scalar field, many aspects of geophysical flows can be described compactly in terms of potential vorticity dynamics. For example, the propagation of Rossby waves and the development of baroclinic instability have traditionally been described in terms of quasigeostrophic potential vorticity dynamics. And in situations in which the quasigeostrophic approximation is inadequate, such as for planetary-scale flows, considerations of potential vorticity dynamics on isentropes (or on isopycnals in the ocean) have proven fruitful (see, e.g., Tung 1986; Rhines and Young 1982). The fact, however, that the potential temperature can fluctuate at boundaries, and the implied inhomogeneous boundary condition for potential vorticity dynamics, complicates considerations of potential vorticity dynamics of flows for which boundary effects are dynamically significant. Bretherton (1966) has shown that the inhomogeneous boundary condition for quasigeostrophic potential vorticity dynamics can be replaced by a homogeneous boundary condition (of constant potential temperature) if the simplification of the boundary condition is compensated by including in the quasigeostrophic potential vorticity a singular surface potential vorticity. The quasigeostrophic surface potential vorticity is proportional to the potential temperature fluctuations at the boundary. Extending Bretherton's ar-

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gumentation, Rhines (1979) has shown that not only surface potential temperature fluctuations, but also the topography of a boundary can be taken into account in a quasigeostrophic surface potential vorticity. Bretherton's and Rhines's generalization of the quasigeostrophic potential vorticity concept has been used to describe the interaction between quasigeostrophic potential vorticity fluctuations in the interior of a flow on the one hand and surface potential temperature fluctuations and/or topographic slopes on the other hand, for example, in unstable baroclinic waves [see Hoskins et al. (1985) and Hallberg and Rhines (2000) for reviews]. Here we present a similar generalization of the potential vorticity concept that allows for the inclusion of boundary effects in the potential vorticity dynamics of arbitrary non-geostrophic flows.

As in quasigeostrophic flows, the inhomogeneous boundary condition for potential vorticity dynamics of arbitrary flows can be replaced by a homogeneous boundary condition if the simplification of the boundary condition is compensated by a generalization of the potential vorticity concept to a sum of the conventional interior potential vorticity and a singular surface potential vorticity. We derive functional forms of the generalized potential vorticity and of its conservation law and discuss non-geostrophic effects described by the generalized potential vorticity conservation law. In order to illustrate how the generalized potential vorticity concept can be used to describe flows for which the quasigeostrophic approximation is inadequate, we demonstrate that this concept can form a basis of theories of lee vortex formation in mesoscale flows past a mountain and of the planetary-scale mean meridional mass flux along isentropes.

Sections 2–4 set up the formal framework of generalized potential vorticity dynamics. Section 2 casts the momentum equation with the help of the thermodynamic equation in the form of field equations in which the potential vorticity and the potential vorticity flux appear as sources of flow fields in the same way in which an electric charge and an electric current appear as sources of fields in electrodynamics. These field equations are the point of departure for the analysis of boundary effects in potential vorticity dynamics. Section 3 derives, by means of techniques from electrodynamics, the functional forms of the generalized potential vorticity and of the generalized potential vorticity flux that replace the conventional interior potential vorticity and the interior potential vorticity flux when the inhomogeneous boundary condition for potential vorticity dynamics is replaced by a homogeneous boundary condition. For the generalized potential vorticity, a conservation law holds that reduces, in the quasigeostrophic limit, to the conservation law for Bretherton's (1966) generalized quasigeostrophic potential vorticity. In section 4, the conservation law for the generalized potential vorticity, derived in coordinate-independent form in section 3, is expanded in isentropic coordinates.

Sections 5 and 6 show how the generalized potential vorticity concept can be used to analyze boundary effects in non-geostrophic mesoscale flows and in the planetary-scale mean meridional circulation. Section 5 discusses the baroclinic induction of generalized potential vorticity at boundaries, a non-geostrophic effect described by the conservation law for generalized potential vorticity. An analysis of a simulated Boussinesq flow demonstrates that the formation of a wake with lee vortices in a flow past a mountain that has no adjacent frictional boundary layer can be described in terms of generalized potential vorticity dynamics and the baroclinic induction of generalized potential vorticity on the mountain surface. A wake with lee vortices can form by separation of a generalized potential vorticity sheet from the mountain surface, similar to the separation of a friction-induced vorticity sheet from an obstacle, except that the generalized potential vorticity sheet can be induced by baroclinicity at the surface. Section 6 discusses the mean budget of generalized potential vorticity in isentropic coordinates. A rigid boundary at which the surface potential temperature fluctuates is a moving boundary in isentropic coordinates, making the entropic flow domain time-dependent. Replacing the conventional potential vorticity concept and the inhomogeneous boundary condition by the generalized potential vorticity concept and a homogeneous boundary condition (of constant potential temperature) allows one to consider a mean potential vorticity budget in a time-independent entropic flow domain. The time-independent entropic flow domain includes the surface layer of isentropes that sometimes intersect the surface. From the mean budget of generalized potential vorticity in the surface layer and in the overlying interior atmosphere, we deduce a balance condition that relates the extratropical mean meridional mass flux along isentropes to eddy fluxes of interior potential vorticity and of surface potential temperature. Since the mean meridional mass flux along isentropes, in concert with radiative processes, sets the thermal stratification of the atmosphere, the generalized potential vorticity concept and the balance condition between the mean meridional mass flux and eddy fluxes can form a basis of theories of the extratropical thermal stratification.

Section 7 summarizes the conclusions. The appendix lists the notation and symbols used in this paper.

The analyses presume an ideal-gas atmosphere with the planet's surface as the only dynamically relevant boundary. However, the concepts and mathematical techniques presented are easily adaptable — for example, to ocean flows with lateral boundaries and with a more complex thermodynamic equation of state, irrespective of the fact that the more complex thermodynamic equation of state of seawater implies that potential vorticity is not necessarily materially conserved in adiabatic and frictionless ocean flows (cf. McDougall 1988).

2. Potential vorticity and potential vorticity flux as sources of flow fields

a. Field equations

The potential vorticity is the pseudoscalar function

$$P = \frac{\boldsymbol{\omega}_{\mathbf{a}} \cdot \boldsymbol{\nabla}\theta}{\rho} \tag{1}$$

of absolute vorticity $\boldsymbol{\omega}_{\mathbf{a}}$, potential temperature θ, and density ρ. The absolute vorticity is the curl $\boldsymbol{\omega}_{\mathbf{a}} = \boldsymbol{\nabla}\times \mathbf{u}_{\mathbf{a}}$ of the three-dimensional absolute velocity $\mathbf{u}_{\mathbf{a}} = \mathbf{u} + \boldsymbol{\Omega}\times\mathbf{r}$, or the sum $\boldsymbol{\omega}_{\mathbf{a}} = \boldsymbol{\omega}_{\mathbf{r}} + 2\boldsymbol{\Omega}$ of the relative vorticity $\boldsymbol{\omega}_{\mathbf{r}} = \boldsymbol{\nabla}\times\mathbf{u}$ and the vorticity $\boldsymbol{\nabla}\times(\boldsymbol{\Omega}\times\mathbf{r}) = 2\boldsymbol{\Omega}$ of a planetary rotation with constant angular velocity $\boldsymbol{\Omega}$. The potential vorticity P is a conserved quantity with a conservation law of the flux form

$$\partial_t(\rho P) + \boldsymbol{\nabla}\cdot(\rho\mathbf{J}) = 0, \tag{2}$$

with a potential vorticity flux[1]

$$\mathbf{J} = \mathbf{u}P - \rho^{-1}Q\boldsymbol{\omega}_{\mathbf{a}} + \rho^{-1}\,\boldsymbol{\nabla}\theta\times\mathbf{F} \tag{3}$$

in which $Q = D\theta/Dt$ is the diabatic heating rate and $\mathbf{F}$ a frictional force per unit mass (Haynes and McIntyre 1987). Diabatic heating Q and frictional forces $\mathbf{F}$ contribute to the potential vorticity flux $\mathbf{J}$ and redistribute potential vorticity within a flow, but in the interior of the flow, they do not create or destroy potential vorticity (Haynes and McIntyre 1987, 1990).

Since the divergence of the absolute vorticity vanishes, $\boldsymbol{\nabla}\cdot\boldsymbol{\omega}_{\mathbf{a}} = 0$, the product $\rho P = \boldsymbol{\omega}_{\mathbf{a}}\cdot\boldsymbol{\nabla}\theta$ of density and potential vorticity is the divergence $\rho P = \boldsymbol{\nabla}\cdot\mathbf{D}$ of a vector field $\mathbf{D}$. The density and potential vorticity determine the vector field $\mathbf{D}$ up to a non-divergent component. For example, one might take as the vector field $\mathbf{D}$ the product $\mathbf{D} = \theta\boldsymbol{\omega}_{\mathbf{a}}$ of potential temperature and absolute vorticity, or the cross product $\mathbf{D} = \mathbf{u}_{\mathbf{a}}\times\boldsymbol{\nabla}\theta$ of absolute velocity and potential temperature gradient. The difference between these two choices for $\mathbf{D}$ is the non-divergent vector field $\boldsymbol{\nabla}\times(\theta\mathbf{u}_{\mathbf{a}})$. For both choices for $\mathbf{D}$, the product ρP is the divergence, and hence the source, of $\mathbf{D}$.

That the product ρP is both a conserved quantity and the divergence of a vector field can be expressed through field equations that make some properties of potential vorticity manifest and that will be convenient in analyzing the role of boundaries in potential vorticity dynamics. Upon substitution of the divergence $\rho P = \boldsymbol{\nabla}\cdot\mathbf{D}$, the conservation law (2) for potential vorticity becomes $\boldsymbol{\nabla}\cdot\big(\partial_t\mathbf{D}+\rho\mathbf{J}\big) = 0$. It follows that the potential vorticity flux density $\rho\mathbf{J}$ has the form

$$\rho\mathbf{J} = -\partial_t\mathbf{D} + \boldsymbol{\nabla}\times\mathbf{H},$$

where $\mathbf{H}$ is the vector potential of the sum $\partial_t\mathbf{D}+\rho\mathbf{J}$. The facts that ρP is both a conserved quantity and the divergence of a vector field $\mathbf{D}$ can thus be expressed through the first two Maxwell equations,

$$\boldsymbol{\nabla}\cdot\mathbf{D} = \rho P, \tag{4a}$$

$$\boldsymbol{\nabla}\times\mathbf{H} - \partial_t\mathbf{D} = \rho\mathbf{J}. \tag{4b}$$

The quantity ρP corresponds to the charge density in electrodynamics, the field $\mathbf{D}$ to the electric displacement field, the field $\mathbf{H}$ to the magnetic field, and the potential vorticity flux density $\rho\mathbf{J}$ to the current density. We refer to the quantity ρP as the potential vorticity density.[2] In the Maxwell equations (4), the potential vorticity P and the potential vorticity flux $\mathbf{J}$ appear as sources of the fields $\mathbf{D}$ and $\mathbf{H}$, just as in electrodynamics charges and currents appear as sources of the electric displacement field and of the magnetic field. The conservation law (2) for potential vorticity follows from the Maxwell equations by adding the time derivative of the first equation (4a) to the divergence of the second equation (4b).

b. Gauge invariance

The potential vorticity P, the potential vorticity flux $\mathbf{J}$, and the density ρ do not determine the fields $\mathbf{D}$ and $\mathbf{H}$ uniquely. The Maxwell equations (4) are invariant under gauge transformations of the form

$$\begin{aligned}\mathbf{D} &\leftarrow \mathbf{D} + \boldsymbol{\nabla}\times\mathbf{A}\\ \mathbf{H} &\leftarrow \mathbf{H} + \partial_t\mathbf{A} + \boldsymbol{\nabla}\psi\end{aligned} \tag{5}$$

where $\mathbf{A}$ is a vector field and ψ a scalar field. Given a potential vorticity P, a potential vorticity flux $\mathbf{J}$, and a density ρ, the fields $\mathbf{D}$ and $\mathbf{H}$ are only determined up to such gauge transformations.

[1]This potential vorticity flux $\mathbf{J}$ differs from the quantity $\rho\mathbf{J}$ that Haynes and McIntyre (1987) call potential vorticity flux by a density factor ρ. We refer to the quantity $\rho\mathbf{J}$ as the potential vorticity flux density. — For a uniqueness property of the potential vorticity flux (3), see Bretherton and Schär (1993).

[2]We prefer the term “potential vorticity density” to Haynes and McIntyre’s (1990) term “potential vorticity substance.” The word substance connotes an independence of other quantities and a subsistence in itself that are not characteristic of the potential vorticity density ρP.

The freedom in the definition of the fields $\mathbf{D}$ and $\mathbf{H}$ can be exploited to find gauges of the fields $\mathbf{D}$ and $\mathbf{H}$ that are amenable to physical interpretation or are convenient in specific contexts.

c. A physically interpretable gauge

With the choice

$$\mathbf{D} = \mathbf{u_a} \times \boldsymbol{\nabla}\theta, \tag{6}$$

the potential vorticity density $\boldsymbol{\nabla}\cdot\mathbf{D} = \rho P$ is proportional to the divergence along isentropes of the absolute angular momentum per unit mass. So the potential vorticity can be interpreted as a source of absolute angular momentum of the flow along isentropes.

A field $\mathbf{H}$ for this choice for $\mathbf{D}$ can be found from the Maxwell equation (4b) by expanding the time derivative $\partial_t\mathbf{D} = \partial_t\mathbf{u_a}\times\boldsymbol{\nabla}\theta + \mathbf{u_a}\times\boldsymbol{\nabla}(\partial_t\theta)$ and by substituting for the velocity derivative $\partial_t\mathbf{u_a} = \partial_t\mathbf{u}$ from the momentum equation

$$\partial_t\mathbf{u} + \boldsymbol{\omega}_\mathbf{a} \times \mathbf{u} = -\frac{1}{\rho}\,\boldsymbol{\nabla}p - \frac{1}{2}\,\boldsymbol{\nabla}\|\mathbf{u}\|^2 - \boldsymbol{\nabla}\Phi + \mathbf{F}$$

and for the potential temperature derivative $\partial_t\theta$ from the thermodynamic equation

$$\partial_t\theta + \mathbf{u}\cdot\boldsymbol{\nabla}\theta = Q. \tag{7}$$

Using the differential $d\hat{h} = \rho^{-1}\,dp + T\,d\hat{s}$ of the specific enthalpy $\hat{h} = c_pT$ and the relation $\boldsymbol{\nabla}\hat{s} = c_p\,\boldsymbol{\nabla}\log\theta$ between gradients of specific entropy $\hat{s}$ and gradients of potential temperature θ, one can cast the momentum equation in the form (cf. Batchelor 1967, chapter 3.5; Schär 1993)

$$\partial_t\mathbf{u} + \boldsymbol{\omega}_\mathbf{a} \times \mathbf{u} = c_pT\,\boldsymbol{\nabla}\log\theta - \boldsymbol{\nabla}B + \mathbf{F} \tag{8}$$

with Bernoulli function

$$B = \frac{1}{2}\|\mathbf{u}\|^2 + c_pT + \Phi. \tag{9}$$

Taking the cross product of the momentum equation (8) and the potential temperature gradient $\boldsymbol{\nabla}\theta$ and using the vector identities $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$ and $\boldsymbol{\nabla}\times(\psi\mathbf{a}) = \boldsymbol{\nabla}\psi\times\mathbf{a} + \psi\,\boldsymbol{\nabla}\times\mathbf{a}$ (for vector fields $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and a scalar field ψ) leads to the Maxwell equation

$$\partial_t\mathbf{D} = -\rho\mathbf{J} + \boldsymbol{\nabla}\times\mathbf{H}$$

with potential vorticity flux (3) and field

$$\mathbf{H} = -B\,\boldsymbol{\nabla}\theta - (\partial_t\theta)\mathbf{u_a}. \tag{10}$$

The explicit time derivative $\partial_t\theta$ in this expression for the field $\mathbf{H}$ could be expanded by substitution from the thermodynamic equation (7); however, with the explicit time derivative $\partial_t\theta$, it is evident that the second term in the expression for the field $\mathbf{H}$ does not appear in a representation of the field in isentropic coordinates. In isentropic coordinates, the field $\mathbf{H}$ is proportional to the Bernoulli function. Since the Bernoulli function is a measure of the specific energy of the moving fluid, the potential vorticity flux can be interpreted as a source (or, because of the minus sign, as a sink) of energy of the flow along isentropes.

In this gauge, then, the potential vorticity P and the potential vorticity flux $\mathbf{J}$ appear as sources of orthogonal fields $\mathbf{D}$ and $\mathbf{H}$ that indicate the absolute angular momentum and the energy of the flow along isentropes. With the potential vorticity flux $\mathbf{J}$ understood as source of a field $\mathbf{H}$ that indicates the energy of the flow along isentropes, it is not as surprising as it might seem that the non-conservative force $\mathbf{F}$ and the diabatic heating rate Q contribute to the potential vorticity flux $\mathbf{J}$.

In a steady state, the Maxwell equation (4b) reduces to $\rho\mathbf{J} = \boldsymbol{\nabla}\times\mathbf{H}$, which becomes, in this gauge, $\rho\mathbf{J} = -\,\boldsymbol{\nabla}\times B\,\boldsymbol{\nabla}\theta = \boldsymbol{\nabla}\theta\times\boldsymbol{\nabla}B$. That is, as noted by Schär (1993), the potential vorticity flux is directed along lines of intersection between surfaces of constant potential temperature θ (isentropes) and surfaces of constant Bernoulli function B. The Bernoulli function B is the streamfunction of the potential vorticity flux $\mathbf{J}$ along isentropes. Some practical implications of this generalization of Bernoulli's theorem are discussed by Schär (1993), Schär and Durran (1997), and, in an oceanic context, by Marshall et al. (2001).

d. A gauge for the analysis of boundary effects

For the analysis of boundary conditions in section 3, the choice

$$\mathbf{D} = \theta\boldsymbol{\omega}_\mathbf{a} \tag{11}$$

is convenient. A field $\mathbf{H}$ for this choice for $\mathbf{D}$ can be found, as above, from the Maxwell equation (4b) by expanding the time derivative $\partial_t\mathbf{D} = (\partial_t\theta)\boldsymbol{\omega}_\mathbf{a} + \theta(\partial_t\boldsymbol{\omega}_\mathbf{a})$ and by substituting for the potential temperature derivative $\partial_t\theta$ from the thermodynamic equation (7) and for the vorticity derivative $\partial_t\boldsymbol{\omega}_\mathbf{a} = \partial_t\boldsymbol{\omega}_\mathbf{r}$ from the vorticity equation

$$\partial_t\boldsymbol{\omega}_\mathbf{r} = \boldsymbol{\nabla}\times\big(\mathbf{u}\times\boldsymbol{\omega}_\mathbf{a} + c_pT\,\boldsymbol{\nabla}\log\theta + \mathbf{F}\big) \tag{12}$$

belonging to the momentum equation (8). Similar algebra to the above leads to the potential vorticity flux (3) and to the field[3]

$$\mathbf{H} = \mathbf{u} \times \theta \boldsymbol{\omega}_\mathbf{a} + c_p T \, \boldsymbol{\nabla}\theta + \theta \mathbf{F}. \tag{13}$$

This functional form of the field $\mathbf{H}$ does not seem to have a simple physical interpretation.

Irrespective of the gauge of the fields $\mathbf{D}$ and $\mathbf{H}$, the Maxwell equations (4) are a way of arranging the momentum equation with the help of the thermodynamic equation such that the existence of a conservation law (2) for potential vorticity is immediately evident. Expressing momentum conservation and potential vorticity conservation through the Maxwell equations makes it possible to use techniques from electrodynamics to analyze the dynamical role of boundaries such as the interface between atmosphere and surface.

3. Generalized potential vorticity in coordinate-independent form

a. Boundary conditions and boundary sources

Since the potential temperature can fluctuate at the surface, the boundary conditions for the fields $\mathbf{D}$ and $\mathbf{H}$ in the Maxwell equations (4) are generally inhomogeneous. These inhomogeneous boundary conditions at the surface, or "immediately above" it, can be replaced by homogeneous boundary conditions "inside" the surface if the simplification of the boundary conditions is compensated by inclusion of suitable boundary sources in the Maxwell equations (see, e.g., Morse and Feshbach 1953, chapter 7). Since the potential vorticity P and the potential vorticity flux $\mathbf{J}$ are sources in the Maxwell equations, the boundary sources can be viewed as boundary contributions to the potential vorticity and to the potential vorticity flux.

For the fields $\mathbf{D}$ and $\mathbf{H}$ in the gauge given by Eqs. (11) and (13), we specify the homogeneous boundary conditions $\mathbf{D}_b = 0$ and $\mathbf{H}_b = 0$ inside the surface. The subscript b denotes quantities inside the surface, and the subscript s will denote quantities immediately above the surface. Taking the fields $\mathbf{D}_b$ and $\mathbf{H}_b$ to be zero inside the surface can be viewed as a consequence of taking the potential temperature θ_b and with it, for consistency with the thermodynamic equation (7) and momentum equation (8), all other flow fields to be zero inside the surface. The homogeneous boundary conditions for the fields $\mathbf{D}$ and $\mathbf{H}$ must be compensated by boundary contributions to the potential vorticity and to the potential vorticity flux.

[3]Alternatively, the frictional term in the potential vorticity flux $\mathbf{J}$ could have been written as $-\rho^{-1}\theta\,\boldsymbol{\nabla}\times\mathbf{F}$, in which case the field $\mathbf{H}$ would not contain a frictional term. However, an advantage of the functional forms (3) and (13) of the potential vorticity flux $\mathbf{J}$ and field $\mathbf{H}$ is that the frictional term $\rho^{-1}\,\boldsymbol{\nabla}\theta\times\mathbf{F}$ in the potential vorticity flux has no cross-isentropic component and is therefore easier to expand in isentropic coordinates than a frictional term of the form $-\rho^{-1}\theta\,\boldsymbol{\nabla}\times\mathbf{F}$ (cf. section 4).

b. Boundary contributions to the potential vorticity

At the interface between two media, the normal component of the electric displacement field has a discontinuity proportional to a surface charge density at the interface (Jackson 1975, section I.5). Analogously, the normal component of the field $\mathbf{D}$ has, at the surface, a discontinuity proportional to a surface density of potential vorticity.

Given the fields $\mathbf{D}_b$ inside the surface and $\mathbf{D}_s$ immediately above the surface with upward normal $\mathbf{n}$, one can compute the surface potential vorticity that is required to force the normal component of the field $\mathbf{D}$ from $\mathbf{n}\cdot\mathbf{D} = \mathbf{n}\cdot\mathbf{D}_b = 0$ inside the surface to $\mathbf{n}\cdot\mathbf{D} = \mathbf{n}\cdot\mathbf{D}_s$ immediately above the surface. Integrating the divergence equation (4a) over an infinitesimally small volume enclosing the surface and using Gauss's theorem to convert the volume integral of the divergence $\boldsymbol{\nabla}\cdot\mathbf{D}$ into a surface integral, one finds that the homogeneous boundary condition $\mathbf{n}\cdot\mathbf{D} = \mathbf{n}\cdot\mathbf{D}_b = 0$ must be compensated by including on the right-hand side of the divergence equation (4a) a source with a surface density of potential vorticity equal to $\mathbf{n}\cdot(\mathbf{D}_s - \mathbf{D}_b) = \mathbf{n}\cdot\mathbf{D}_s$ (see, e.g., Jackson 1975, section I.5; Morse and Feshbach 1953, chapter 7). One can think of this surface density of potential vorticity as the across-surface integral of the potential vorticity density ρS that belongs to the singular surface potential vorticity

$$S = \frac{\mathbf{n}\cdot\mathbf{D}_s}{\rho}\,\delta(z - z_s).$$

The surface potential vorticity S is concentrated in a delta-function potential vorticity sheet on the surface at $z = z_s(x, y)$. The choice $\mathbf{D} = \theta\boldsymbol{\omega}_\mathbf{a}$ for the field $\mathbf{D}$ leads to the surface potential vorticity

$$S = \frac{\boldsymbol{\omega}_\mathbf{a}\cdot\mathbf{n}}{\rho}\,\theta\,\delta(z - z_s). \tag{14}$$

The divergence equation (4a) with inhomogeneous boundary condition $\mathbf{n}\cdot\mathbf{D} = \mathbf{n}\cdot\mathbf{D}_s$ and potential vorticity P is equivalent to a divergence equation (4a) with homogeneous boundary condition $\mathbf{n}\cdot\mathbf{D} = \mathbf{n}\cdot\mathbf{D}_b = 0$ and generalized potential vorticity

$$P_g = P + S. \tag{15}$$

Replacing the inhomogeneous boundary condition for the field $\mathbf{D}$ by a homogeneous boundary condition is compensated by putting an idealized potential vorticity sheet with surface potential vorticity S on the surface.

c. Boundary contributions to the potential vorticity flux

At the interface between two media, the tangential component of the magnetic field has a discontinuity proportional to a current density at the interface (Jackson 1975, section I.5). Analogously, the tangential component of the field $\mathbf{H}$ has, at the surface, a discontinuity proportional to a surface density of potential vorticity flux.

The surface flux of potential vorticity that is required to force the tangential component $\mathbf{n} \times \mathbf{H}$ of the field $\mathbf{H}$ from $\mathbf{n} \times \mathbf{H} = \mathbf{n} \times \mathbf{H}_b = 0$ inside the surface to $\mathbf{n} \times \mathbf{H} = \mathbf{n} \times \mathbf{H}_s$ immediately above the surface can be found by integrating the second Maxwell equation $\boldsymbol{\nabla} \times \mathbf{H} - \partial_t \mathbf{D} = \rho \mathbf{J}$ over an area perpendicular to and including the surface. In the limit of an infinitesimally small area, the area integral of the time derivative $\partial_t \mathbf{D}$ vanishes because $\partial_t \mathbf{D}$ is finite at the surface. Using Stokes's theorem to convert the area integral of the curl $\boldsymbol{\nabla} \times \mathbf{H}$ into a loop integral and choosing the area of integration such that those segments of the loop integral that are perpendicular to the surface are infinitesimally small, one finds that the homogeneous boundary condition $\mathbf{n} \times \mathbf{H} = \mathbf{n} \times \mathbf{H}_b = 0$ must be compensated by including on the right-hand side of the Maxwell equation (4b) a source with a surface density of potential vorticity flux equal to $\mathbf{n} \times (\mathbf{H}_s - \mathbf{H}_b) = \mathbf{n} \times \mathbf{H}_s$ (see Jackson 1975, section I.5). Analogous to the line of reasoning that led to the surface potential vorticity, one can think of the surface density of potential vorticity flux as the across-surface integral of the potential vorticity flux density $\rho \mathbf{K}$ that belongs to the singular potential vorticity flux

$$\mathbf{K} = \frac{\mathbf{n} \times \mathbf{H}_s}{\rho}\, \delta(z - z_s).$$

This surface potential vorticity flux is concentrated on the surface and has only components tangential to the surface. Substituting for the field $\mathbf{H}$ from the above-derived expression (13) and using the fact that, at the surface, the normal component $\mathbf{n} \cdot \mathbf{u}$ of the velocity $\mathbf{u}$ vanishes, one obtains the surface potential vorticity flux

$$\mathbf{K} = \mathbf{u} S + \mathbf{K}_b + \mathbf{K}_F \tag{16a}$$

with baroclinic component

$$\mathbf{K}_b = \rho^{-1} c_p T\, (\mathbf{n} \times \boldsymbol{\nabla}\theta)\, \delta(z - z_s) \tag{16b}$$

and frictional component

$$\mathbf{K}_F = \rho^{-1} \theta\, (\mathbf{n} \times \mathbf{F})\, \delta(z - z_s). \tag{16c}$$

The surface potential vorticity flux $\mathbf{K}$ consists of an advective component $\mathbf{u}S$, of a baroclinic component $\mathbf{K}_b$ that is directed along isentropes lying in the surface, and of a frictional component $\mathbf{K}_F$.[4] The baroclinic component $\mathbf{K}_b$ has its origin in the contribution of the baroclinicity vector $\rho^{-2}\, \boldsymbol{\nabla}\rho \times \boldsymbol{\nabla} p$ to the vector potential $\mathbf{H}$. With a no-slip boundary condition at the surface, the velocity $\mathbf{u}$ along the surface vanishes, and the surface potential vorticity flux $\mathbf{K}$ consists only of the non-advective components $\mathbf{K}_b$ and $\mathbf{K}_F$. The advective component of the surface potential vorticity flux $\mathbf{K}$ might, nevertheless, be of practical relevance. For example, numerical atmosphere models typically use not a no-slip boundary condition, but a drag-law boundary condition, so the advective component of the surface potential vorticity flux might not be negligible in a generalized potential vorticity budget of such a model. The advective component of the surface potential vorticity flux is proportional to the surface heat flux, and horizontal heat fluxes are significant down to the lowest levels of typical general circulation models and down to the lowest atmospheric levels for which observational data are available [cf. Held and Schneider (1999) and section 6e].

The Maxwell equation (4b) with inhomogeneous boundary condition $\mathbf{n} \times \mathbf{H} = \mathbf{n} \times \mathbf{H}_s$ and potential vorticity flux $\mathbf{J}$ is equivalent to a Maxwell equation (4b) with homogeneous boundary condition $\mathbf{n} \times \mathbf{H} = \mathbf{n} \times \mathbf{H}_b = 0$ and generalized potential vorticity flux

$$\mathbf{J}_g = \mathbf{J} + \mathbf{K}. \tag{17}$$

Replacing the inhomogeneous boundary condition for the field $\mathbf{H}$ by a homogeneous boundary condition is compensated by the added flux $\mathbf{K}$ in the idealized potential vorticity sheet on the surface.

d. Conservation of generalized potential vorticity

The generalized potential vorticity P_g and the generalized potential vorticity flux $\mathbf{J}_g$ contain the boundary contributions to the potential vorticity budget. The original Maxwell equations (4) with inhomogeneous boundary conditions at the surface and with the interior potential vorticity P and the interior potential vorticity flux $\mathbf{J}$ as sources are equivalent to Maxwell equations with homogeneous boundary conditions inside the surface and with the generalized potential vorticity P_g and the generalized potential vorticity flux $\mathbf{J}_g$ as sources. Adding the time derivative of the first Maxwell equation (4a) to the divergence of the second Maxwell equation (4b) yields the conservation law

$$\partial_t(\rho P_g) + \boldsymbol{\nabla} \cdot (\rho \mathbf{J}_g) = 0 \tag{18}$$

[4] If the frictional term in the interior potential vorticity flux $\mathbf{J}$ would have been written as $-\rho^{-1}\theta\, \boldsymbol{\nabla} \times \mathbf{F}$, the field $\mathbf{H}$ and, with it, the surface potential vorticity flux $\mathbf{K}$ would not contain frictional components (cf. footnote 3).

for the generalized potential vorticity P_g.

The conservation law (18) for the generalized potential vorticity is similar to the conservation law (2) for the interior potential vorticity. In contrast to the interior potential vorticity, however, the generalized potential vorticity is not, in general, materially conserved in adiabatic and frictionless flows. The baroclinic component (16b) of the surface potential vorticity flux can redistribute generalized potential vorticity non-advectively along the surface.

Nevertheless, the integral of the generalized potential vorticity density ρP_g over the volume of the atmosphere (or over a suitable control volume) is conserved. Integrating the conservation law (18) over the volume V of the atmosphere and using Gauss's theorem with the boundary condition $\mathbf{n}\cdot(\rho\mathbf{J}_g) = 0$ inside the surface and with the assumption that the generalized potential vorticity flux density $\rho\mathbf{J}_g$ vanish at the top of the atmosphere, one obtains

$$\partial_t \int_V \rho P_g \, d\mathbf{x} = 0,$$

or, equivalently,

$$\partial_t \int_V \rho P \, d\mathbf{x} = -\partial_t \int_V \rho S \, d\mathbf{x}.$$

Any increase in the volume-integrated interior potential vorticity density ρP is compensated by a decrease in the volume-integrated surface potential vorticity density ρS, and vice versa. The conservation law (2) for the interior potential vorticity implies that the volume-integrated interior potential vorticity density ρP can change only if the integral of the normal component

$$\mathbf{n}\cdot(\rho\mathbf{J}) = -Q(\mathbf{n}\cdot\boldsymbol{\omega}_{\mathbf{a}}) + \mathbf{n}\cdot(\boldsymbol{\nabla}\theta\times\mathbf{F})$$

of the interior potential vorticity flux density over the surface area A is nonzero,

$$\begin{aligned}\partial_t \int_V \rho P \, d\mathbf{x} \\ &= -\int_A \Big[Q\big(\mathbf{n}\cdot\boldsymbol{\omega}_{\mathbf{a}}\big) - \mathbf{n}\cdot\big(\boldsymbol{\nabla}\theta\times\mathbf{F}\big)\Big]\, dA \\ &= -\partial_t \int_V \rho S \, d\mathbf{x}. \qquad (19)\end{aligned}$$

Since the normal component of the velocity vanishes at the surface, only diabatic and frictional processes at the surface can effect changes in the volume-integrated interior potential vorticity density. Given that changes in volume-integrated interior potential vorticity density are compensated by opposing changes in volume-integrated surface potential vorticity density, diabatic and frictional processes at the surface can be viewed as converting surface potential vorticity into interior potential vorticity, and vice versa. The surface integral of the normal component $\mathbf{n}\cdot(\rho\mathbf{J}_g) = \mathbf{n}\cdot(\rho\mathbf{J})$ of the potential vorticity flux density indicates the conversion rate of volume-integrated surface potential vorticity density into volume-integrated interior potential vorticity density.

e. Alternative generalized potential vorticity functionals

The functional forms of the generalized potential vorticity $P_g = P + S$ and of the generalized potential vorticity flux $\mathbf{J}_g = \mathbf{J} + \mathbf{K}$ are not unique because the functional forms of the surface potential vorticity S and of the surface potential vorticity flux $\mathbf{K}$ depend on the gauge of the fields $\mathbf{D}$ and $\mathbf{H}$.

We chose the gauge given by Eqs. (11) and (13) because, in this gauge, the functional forms of the surface potential vorticity (14) and of the surface potential vorticity flux (16) resemble the functional forms of the interior potential vorticity (1) and of the interior potential vorticity flux (3). For example, the surface potential vorticity (14) is proportional to the absolute vorticity component $\boldsymbol{\omega}_{\mathbf{a}}\cdot\mathbf{n}$ normal to the surface, while the interior potential vorticity (1) is proportional to the absolute vorticity component $\boldsymbol{\omega}_{\mathbf{a}}\cdot\boldsymbol{\nabla}\theta$ normal to isentropes. And like the interior potential vorticity flux (3), the surface potential vorticity flux (16) contains an advective component, which legitimizes its interpretation as a flux.

Alternatively, however, we could have chosen a gauge in which, for example, a field $\mathbf{D}' = \theta'\boldsymbol{\omega}_{\mathbf{a}}$ is defined with the potential temperature fluctuation $\theta' = \theta - \theta_0$ about a constant reference potential temperature θ_0 in place of the absolute potential temperature θ. Such a gauge suggests itself when a reference potential temperature is given, such as is the case in Boussinesq flows (see below). One obtains the alternative gauge $\mathbf{D}'$ and $\mathbf{H}'$ from the gauge given by Eqs. (11) and (13) by a transformation of the form (5) with $\mathbf{A} = -\theta_0\mathbf{u}_{\mathbf{a}}$. Replacing the inhomogeneous boundary conditions $\mathbf{n}\cdot\mathbf{D}' = \mathbf{n}\cdot\mathbf{D}'_s$ and $\mathbf{n}\times\mathbf{H}' = \mathbf{n}\times\mathbf{H}'_s$ by homogeneous boundary conditions $\mathbf{n}\cdot\mathbf{D}' = \mathbf{n}\cdot\mathbf{D}'_b = 0$ and $\mathbf{n}\times\mathbf{H}' = \mathbf{n}\times\mathbf{H}'_b = 0$ in this alternative gauge results in a surface potential vorticity $S' = \theta'/\theta\, S$ and a surface potential vorticity flux $\mathbf{K}' = \theta'/\theta\,\mathbf{K}$. All of the above statements about the conservation of generalized potential vorticity and about the structure of the surface potential vorticity and the surface potential vorticity flux remain valid if the surface potential vorticity S and the surface potential vorticity flux $\mathbf{K}$ are replaced by the alternative functionals S' and $\mathbf{K}'$. This arbitrariness in the definition of the surface potential vorticity and of the surface potential vorticity flux means that their absolute values by themselves carry no dynamical significance.

Other gauges are possible and may be convenient in some contexts (see section 6e for an example). However, the conservation law (18) for the generalized potential vorticity does not depend on the gauge chosen. The gauge invariance of the Maxwell equations (4) translates into gauge invariance of the conservation law (18) for generalized potential vorticity.

In the derivations so far, neither the continuity equation nor the hydrostatic approximation has been used. Conservation of generalized potential vorticity holds quite generally for compressible and non-hydrostatic flows. For adiabatic and frictionless Boussinesq flows and for quasigeostrophic flows over a flat surface, the expressions for the surface potential vorticity and the surface potential vorticity flux simplify.

f. Adiabatic and frictionless Boussinesq flows

The surface potential vorticity S and the surface potential vorticity flux $\mathbf{K}$ for adiabatic and frictionless Boussinesq flows can be derived in a similar manner as the above expressions for the surface potential vorticity (14) and the surface potential vorticity flux (16). In the Boussinesq approximation, the density ρ in the potential vorticity (1) is taken to be equal to a constant reference density ρ_0, and the potential vorticity is defined with the potential temperature fluctuation $\theta' = \theta - \theta_0$ about a constant reference potential temperature θ_0 in place of the absolute potential temperature θ,

$$P = \frac{\boldsymbol{\omega}_{\mathbf{a}} \cdot \boldsymbol{\nabla}\theta'}{\rho_0}. \tag{20}$$

Correspondingly, the surface potential vorticity becomes

$$S = \frac{\boldsymbol{\omega}_{\mathbf{a}} \cdot \mathbf{n}}{\rho_0}\, \theta'\, \delta(z - z_s). \tag{21}$$

The surface potential vorticity flux for adiabatic and frictionless flows consists of the advective component $\mathbf{u}S$ and of a baroclinic component $\mathbf{K}_b$. Which form the baroclinic component $\mathbf{K}_b$ of the surface potential vorticity flux takes for Boussinesq flows can be seen by going back to the derivation of the surface potential vorticity flux in the general case. The baroclinic component has its origin in the baroclinicity vector $-\boldsymbol{\nabla}\times \rho^{-1}\,\boldsymbol{\nabla}p$. The baroclinic component resulted from writing the term $-\theta(\boldsymbol{\nabla}\times \rho^{-1}\,\boldsymbol{\nabla}p)$ in the expansion of the time derivative $\partial_t\mathbf{D}$ as the curl of the vector field $c_pT\,\boldsymbol{\nabla}\theta$, making this field $c_pT\,\boldsymbol{\nabla}\theta$ part of the field $\mathbf{H}$ [Eq. (13)], and taking the tangential component $\mathbf{n}\times\mathbf{H}$ to determine the surface potential vorticity flux. In the Boussinesq approximation, the baroclinicity vector is $\boldsymbol{\nabla}\times \frac{g\theta'}{\theta_0}\,\mathbf{k}$, with vertical unit vector $\mathbf{k}$, and the term $-\theta(\boldsymbol{\nabla}\times \rho^{-1}\,\boldsymbol{\nabla}p)$ in the general case becomes $\theta'\big(\boldsymbol{\nabla}\times \frac{g\theta'}{\theta_0}\,\mathbf{k}\big) = \boldsymbol{\nabla}\times \frac{g\theta'^2}{2\theta_0}\,\mathbf{k}$. Consequently, the baroclinic component of the surface potential vorticity flux becomes

$$\mathbf{K}_b = \frac{1}{\rho_0}\,\frac{g\theta'^2}{2\theta_0}(\mathbf{n}\times\mathbf{k})\,\delta(z - z_s). \tag{22}$$

The baroclinic component of the surface potential vorticity flux is quadratic in potential temperature fluctuations θ' and hence would not appear in a linearized Boussinesq system. Instead of being directed along isentropes lying in the surface, as the baroclinic component (16b) in the general case, the baroclinic component of the Boussinesq surface potential vorticity flux is directed along lines of constant surface elevation. At a flat surface ($\mathbf{n}=\mathbf{k}$), the baroclinic component vanishes, and the surface potential vorticity flux reduces to the advective flux $\mathbf{K} = \mathbf{u}S$. Since the Boussinesq approximation is often an adequate approximation for atmospheric flows near the surface — say, within the planetary boundary layer — the vanishing of the baroclinic component $\mathbf{K}_b$ of the surface potential vorticity flux for Boussinesq flows over a flat surface suggests that this component is only important if topography exerts a significant influence on the flow. The baroclinic component $\mathbf{K}_b$ of the surface potential vorticity flux and topographic effects will be discussed in more detail in section 5.

g. Quasigeostrophic Boussinesq flows over flat surface

Taking the quasigeostrophic limit of the Maxwell equations for Boussinesq flows, one finds that, for quasigeostrophic Boussinesq flows over a flat surface ($z_s = 0$), the normal component $\boldsymbol{\omega}_{\mathbf{a}}\cdot\mathbf{n}$ of the absolute vorticity in the surface potential vorticity (21) must be approximated by a constant reference value f_0 of the Coriolis parameter f, so that the surface potential vorticity becomes

$$S = \frac{f_0}{\rho_0}\,\theta'\,\delta(z).$$

This surface potential vorticity corresponds to that boundary contribution to the quasigeostrophic potential vorticity with which Bretherton (1966) replaced an inhomogeneous thermodynamic boundary condition.

The baroclinic component (22) of the surface potential vorticity flux vanishes for Boussinesq flows over a flat surface, and, for adiabatic and frictionless quasigeostrophic flows, the surface potential vorticity flux reduces to the advective flux

$$\mathbf{K} = \mathbf{u}_g S = \frac{f_0}{\rho_0}\mathbf{u}_g\theta'\,\delta(z),$$

where the advecting velocity $\mathbf{u}_g$ is the geostrophic velocity. For quasigeostrophic Boussinesq flows over a flat surface, the surface potential vorticity flux is proportional to the geostrophic surface heat flux. This surface potential vorticity flux corresponds to the boundary contribution to the quasigeostrophic potential vorticity flux discussed by Bretherton (1966). Bretherton's insight that "any flow with potential temperature variations over a horizontal rigid plane boundary may be considered equivalent to a flow without such variations, but with a concentration of potential vorticity very close to the boundary" (p. 329) generalizes from quasigeostrophic flows to arbitrary atmospheric flows: any flow with potential temperature variations over a rigid surface may be considered equivalent to a flow without such variations, but with a concentrated surface potential vorticity (14) and a concentrated surface potential vorticity flux (16) immediately above the surface.

4. Generalized potential vorticity in isentropic coordinates

We assume that for each time t and at each point in the (x, y)-plane, the potential temperature θ is a strictly monotonic function of height z, so that the instantaneous thermal stratification is everywhere statically stable ($\partial_z \theta > 0$) and the potential temperature can be used as the vertical coordinate in an isentropic coordinate system. We adopt the hydrostatic approximation and carry out the analysis within the framework of the primitive equations.

We will determine the generalized potential vorticity and the components of the generalized potential vorticity flux in isentropic coordinates by expanding the coordinate-independent expressions of sections 2 and 3 in isentropic coordinates. Expressions for the interior potential vorticity (1) and for the interior potential vorticity flux (3) in isentropic coordinates are well-known; they are usually derived from the equations of motion in isentropic coordinates [see, e.g., Salmon (1998, chapter 2.18) or Andrews et al. (1987, chapter 3.8)]. The technique of expanding the coordinate-independent expressions in isentropic coordinates has the advantage of being applicable to the singular surface potential vorticity and its flux, without it being necessary to go back to the equations of motion to deduce the representation of these quantities in isentropic coordinates.

Isentropic coordinates are non-orthogonal, so contravariant and covariant vector components must be distinguished (see, e.g., Arfken 1985, chapter 3). We use the notation $\left(a^x,\ a^y,\ a^\theta\right)_\theta$ for the contravariant components of a vector $\mathbf{a}$ in isentropic coordinates. The contravariant horizontal components $a^x = \mathbf{a} \cdot \mathbf{i}$ and $a^y = \mathbf{a} \cdot \mathbf{j}$ are equal to the local Cartesian components of the vector $\mathbf{a}$, the local Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$ being directed eastward and northward.[5] The contravariant cross-isentropic component $a^\theta = \mathbf{a} \cdot \boldsymbol{\nabla}\theta$ is the scalar product of the vector $\mathbf{a}$ and the potential temperature gradient $\boldsymbol{\nabla}\theta$.

a. *Generalized potential vorticity*

The generalized potential vorticity takes a particularly simple form in isentropic coordinates. In the primitive equations, the planetary vorticity $2\boldsymbol{\Omega}$ is approximated by its local vertical component $f\mathbf{k}$, and in the hydrostatic approximation, horizontal derivatives of the vertical velocity in the relative vorticity $\boldsymbol{\omega}_\mathbf{r} = \boldsymbol{\nabla} \times \mathbf{u}$ are neglected compared with vertical derivatives of the horizontal velocity. The absolute vorticity hence becomes the sum $\boldsymbol{\omega}_\mathbf{a} = f\mathbf{k} + \boldsymbol{\nabla} \times \mathbf{v}$ of the planetary vorticity $f\mathbf{k}$ and the relative vorticity $\boldsymbol{\nabla} \times \mathbf{v}$ of the horizontal flow $\mathbf{v} = (u, v, 0)$. The relative vorticity of the horizontal flow can be represented in isentropic coordinates as

$$\boldsymbol{\nabla} \times \mathbf{v} = h^{-1}\left(-\partial_\theta v,\ \partial_\theta u,\ \partial_x v - \partial_y u\right)_\theta, \qquad (23)$$

where the horizontal derivatives ∂_x and ∂_y are to be understood as derivatives along isentropes, and the scale factor

$$h = \partial_\theta z, \qquad (24)$$

an inverse measure of static stability, is the Jacobian $h = \partial(x, y, z)/\partial(x, y, \theta)$ of the transformation from Cartesian coordinates to isentropic coordinates. By the representation (23) of the relative vorticity in isentropic coordinates, the interior potential vorticity density $\rho P = \boldsymbol{\omega}_\mathbf{a} \cdot \boldsymbol{\nabla}\theta$ — the contravariant cross-isentropic component of the absolute vorticity — is $\boldsymbol{\omega}_\mathbf{a} \cdot \boldsymbol{\nabla}\theta = h^{-1}(f + \zeta_\theta)$, where $\zeta_\theta = \partial_x v - \partial_y u$ is the relative vorticity of the horizontal flow $\mathbf{v}$ along isentropes. In the hydrostatic approximation, the density is $\rho = -g^{-1}\partial_z p$, and the product of density ρ and scale factor h is the isentropic density

$$\rho_\theta = \rho h = -g^{-1}\partial_\theta p.$$

Combining the density ρ and the scale factor h in the potential vorticity density $\boldsymbol{\omega}_\mathbf{a} \cdot \boldsymbol{\nabla}\theta = h^{-1}(f + \zeta_\theta)$ yields the well-known interior potential vorticity

$$P = \frac{f + \zeta_\theta}{\rho_\theta}\, \mathcal{H}(\theta - \theta_s). \qquad (25)$$

[5] As horizontal coordinates, we use local Cartesian coordinates in what follows. The transformation of the horizontal coordinates from local Cartesian coordinates to spherical coordinates is straightforward.

The step function

$$\mathcal{H}(\theta-\theta_s) = \begin{cases} 1 & \text{if } \theta > \theta_s \\ 0 & \text{if } \theta < \theta_s \end{cases}$$

indicates that the interior potential vorticity P contributes to the generalized potential vorticity P_g only above the surface, on isentropes with potential temperature θ greater than the surface potential temperature $\theta_s(x, y, t)$.

An isentropic-coordinate representation of the boundary contribution S to the generalized potential vorticity $P_g = P + S$ can be found in a similar way. Under the assumption of static stability, the delta-function $\delta(z - z_s)$ transforms according to $\delta(z - z_s) = h^{-1}\,\delta(\theta - \theta_s)$, where the scale factor h is to be evaluated immediately above the surface. Combining the density ρ in the surface potential vorticity (14) with the scale factor h from the transformation of the delta-function yields the isentropic-coordinate expression

$$S = \frac{\boldsymbol{\omega}_{\mathbf{a}} \cdot \mathbf{n}}{\rho_\theta}\,\theta\,\delta(\theta - \theta_s) \tag{26}$$

for the surface potential vorticity.

Within the approximations of the primitive equations and under the assumption of static stability, the generalized potential vorticity (15) is the sum of the interior potential vorticity (25) and the surface potential vorticity (26) in isentropic coordinates.

b. Generalized potential vorticity flux

The generalized potential vorticity flux in isentropic coordinates can likewise be found by expanding the vectors and differential operators of the coordinate-independent interior potential vorticity flux (3) and surface potential vorticity flux (16). The isentropic-coordinate representation of the term $\rho^{-1}Q\boldsymbol{\omega}_{\mathbf{a}}$ in the interior potential vorticity flux (3) follows by expanding the relative vorticity $\boldsymbol{\nabla}\times\mathbf{v}$ with the help of the expression (23); the frictional force per unit mass $\mathbf{F}$ is assumed to have only horizontal components F^x and F^y; and products of the scale factor h and the density ρ are combined to the isentropic density ρ_θ. The cross-isentropic components of the advective flux $\mathbf{u}P$ and of the diabatic term $\rho^{-1}Q\boldsymbol{\omega}_{\mathbf{a}}$ cancel because, by the thermodynamic equation (7), the contravariant cross-isentropic component $\mathbf{u}\cdot\boldsymbol{\nabla}\theta$ of the velocity is the heating rate Q, and the contravariant cross-isentropic component $\boldsymbol{\omega}_{\mathbf{a}}\cdot\boldsymbol{\nabla}\theta$ of the absolute vorticity is the potential vorticity density ρP. Combining all terms, one finds the well-known interior potential vorticity flux (cf. Haynes and McIntyre 1987)

$$\mathbf{J} = (u,\ v,\ 0)_\theta P + \mathbf{J}_Q + \mathbf{J}_F \tag{27a}$$

with diabatic flux

$$\mathbf{J}_Q = \rho_\theta^{-1} Q\,(\partial_\theta v,\ -\partial_\theta u,\ 0)_\theta\,\mathcal{H}(\theta-\theta_s) \tag{27b}$$

and frictional flux

$$\mathbf{J}_F = \rho_\theta^{-1}\left(-F^y,\ F^x,\ 0\right)_\theta \mathcal{H}(\theta-\theta_s). \tag{27c}$$

Even in the presence of diabatic heating and friction, the interior potential vorticity flux has no cross-isentropic component. Therefore, the impermeability theorem holds: interior potential vorticity can only be redistributed along isentropes but cannot be transferred across isentropes (Haynes and McIntyre 1987).

In order to represent the surface potential vorticity flux (16) in isentropic coordinates, we use for the unit normal vector at the surface $z = z_s(x, y)$ the explicit representation

$$\mathbf{n} = \mu\,\boldsymbol{\nabla}(z - z_s) = \mu(\mathbf{k} - \boldsymbol{\nabla} z_s)$$

with normalization factor

$$\mu = \left(1 + \|\boldsymbol{\nabla} z_s\|^2\right)^{-1/2}.$$

The hydrostatic approximation is only justifiable if the horizontal scale of the topography $z_s(x, y)$ is much greater than the vertical scale, such that $\mu \approx 1$. For consistency with the hydrostatic approximation, we should set the normalization factor μ equal to one. But with the understanding that the hydrostatic approximation would be inappropriate if the normalization factor μ were significantly less than one, we retain the normalization factor μ in the following equations as a marker of where topographic effects can play a role.

With the explicit representation of the normal vector $\mathbf{n}$, the surface potential vorticity flux can be expanded in isentropic coordinates term-by-term. The delta-functions in the surface potential vorticity flux are transformed in the same way as above: $\delta(z - z_s) = h^{-1}\,\delta(\theta - \theta_s)$. And horizontal derivatives of the potential temperature θ at constant height z are transformed into horizontal derivatives of the height z at constant potential temperature θ by means of the relation $\partial_{x_i}\theta|_z = -h^{-1}\,\partial_{x_i} z|_\theta$ for $x_i = x, y$. One obtains the surface potential vorticity flux

$$\mathbf{K} = (u,\ v,\ Q)_\theta S + \mathbf{K}_b + \mathbf{K}_F \tag{28a}$$

with baroclinic component

$$\mathbf{K}_b = \frac{\mu E}{\rho_\theta}\left(-\theta_s^y,\ \theta_s^x,\ 0\right)_\theta \theta\,\delta(\theta-\theta_s) \tag{28b}$$

and frictional component

$$\mathbf{K}_F = \frac{\mu}{\rho_\theta}\left(-F^y,\ F^x,\ F^x\theta_s^y - F^y\theta_s^x\right)_\theta \theta\,\delta(\theta-\theta_s), \tag{28c}$$

where

$$\theta_s^x = -h^{-1}\partial_x(z-z_s) \quad \text{and} \quad \theta_s^y = -h^{-1}\partial_y(z-z_s) \tag{29}$$

are the derivatives of the surface potential temperature $\theta_s(x,y,t)$ with respect to x and y. In the baroclinic component (28b), the specific enthalpy c_pT has been written as the product $c_pT = \theta E$ of potential temperature θ and Exner function $E = c_p(p/p_0)^\kappa$, p_0 being a constant reference pressure.

Within the approximations of the primitive equations and under the assumption of static stability, the generalized potential vorticity flux (17) is the sum of the interior potential vorticity flux (27) and the surface potential vorticity flux (28) in isentropic coordinates. Since both the advective component (28a) and the frictional component (28c) of the surface potential vorticity flux have cross-isentropic components, the impermeability theorem does not hold for the generalized potential vorticity flux in the gauge chosen here. The generalized potential vorticity of the gauge chosen here can be transferred across isentropes by diabatic heating and friction at the surface.

c. *Conservation of generalized potential vorticity*

The conservation law (18) for the generalized potential vorticity becomes in isentropic coordinates

$$\partial_t(\rho_\theta P_g) + \partial_x\left(\rho_\theta J_g^x\right) + \partial_y\left(\rho_\theta J_g^y\right) + \partial_\theta\left(\rho_\theta J_g^\theta\right) = 0, \tag{30}$$

where $\left(J_g^x,\ J_g^y,\ J_g^\theta\right)_\theta$ are the contravariant components of the generalized potential vorticity flux $\mathbf{J}_g$.[6]

[6]The conservation law (30) results from the representation

$$\boldsymbol{\nabla}\cdot\mathbf{a} = \frac{1}{h}\left[\partial_x\left(ha^x\right) + \partial_y\left(ha^y\right) + \partial_\theta\left(ha^\theta\right)\right]$$

of the divergence of a vector $\mathbf{a} = \left(a^x,\ a^y,\ a^\theta\right)_\theta$ in isentropic coordinates (cf. Arfken 1985, chapter 3.9). Since the scale factor h can be viewed as the Jacobian $h = \partial(x,y,z,t)/\partial(x,y,\theta,t)$ of the four-dimensional transformation from (x,y,z,t)-coordinates to (x,y,θ,t)-coordinates, and since it is the Jacobian h of the transformation from Cartesian to isentropic coordinates that appears in the representation of the divergence in isentropic coordinates, the explicit time derivative $\partial_t(\rho P)$ can be viewed as being part of a four-dimensional divergence operator and can, like the space derivatives, be written as $h^{-1}\partial_t(h\rho P) = h^{-1}\partial_t(\rho_\theta P)$; hence the conservation law (30) in isentropic coordinates.

5. Baroclinic induction of generalized potential vorticity

In adiabatic and frictionless flows, the surface potential vorticity flux (28) is the sum $\mathbf{K} = (u,\ v,\ 0)_\theta S + \mathbf{K}_b$ of the advective component $(u,\ v,\ 0)_\theta S$ and the baroclinic component $\mathbf{K}_b$. The presence of the non-advective baroclinic component $\mathbf{K}_b$ of the surface potential vorticity flux implies that surface potential vorticity, and through it generalized potential vorticity, can be induced baroclinically.

a. *Origin of baroclinic component of surface potential vorticity flux*

The baroclinic component $\mathbf{K}_b$ of the surface potential vorticity flux arises because of differences between the interior potential vorticity $(f+\zeta_\theta)/\rho_\theta$ and the quantity $(\boldsymbol{\omega}_\mathbf{a}\cdot\mathbf{n})/\rho_\theta$ to which the surface potential vorticity (26) is proportional. Denoting the relative vorticity component perpendicular to the surface by $\zeta_\sigma = \boldsymbol{\omega}_\mathbf{r}\cdot\mathbf{n}$, one can write $(\boldsymbol{\omega}_\mathbf{a}\cdot\mathbf{n})/\rho_\theta = (\mu f + \zeta_\sigma)/\rho_\theta$. In the hydrostatic approximation, the normalization factor μ is equal to one, so that the interior potential vorticity $(f+\zeta_\theta)/\rho_\theta$ and the quantity $(\boldsymbol{\omega}_\mathbf{a}\cdot\mathbf{n})/\rho_\theta = (f+\zeta_\sigma)/\rho_\theta$ differ only by the relative vorticity factors: the interior potential vorticity

$$P = \frac{f+\zeta_\theta}{\rho_\theta}\,\mathcal{H}(\theta-\theta_s)$$

contains the relative vorticity ζ_θ of the flow along isentropes; the surface potential vorticity

$$S = \frac{f+\zeta_\sigma}{\rho_\theta}\,\theta\,\delta(\theta-\theta_s)$$

contains the relative vorticity ζ_σ of the flow along the surface. Because of this difference in the relative vorticity factors, the surface potential vorticity flux has a non-advective baroclinic component in adiabatic and frictionless flows, while the interior potential vorticity flux is purely advective in such flows.

The conservation law for the interior potential vorticity P is the regular part of the conservation law (30) for the generalized potential vorticity $P_g = P + S$. The regular part of the conservation law (30) describes the time evolution of the isentropic relative vorticity ζ_θ. In the absence of diabatic heating and friction, the interior potential vorticity flux (27) reduces to an advective flux along isentropes, so the interior potential vorticity $(f+\zeta_\theta)/\rho_\theta$ is materially conserved and is, in particular, materially conserved in adiabatic and frictionless flows along isentropes at or immediately above the surface. In contrast, the conservation law for the surface potential vorticity

S is the singular part of the conservation law (30) for the generalized potential vorticity. The singular part of the conservation law (30) describes the time evolution of the relative vorticity ζ_σ of the flow along isentropes $\theta = \theta_s(x, y, t)$ at the surface. Even in the absence of diabatic heating and friction, the surface potential vorticity flux (28) contains the non-advective baroclinic component $\mathbf{K}_b$, so the quantity $(f + \zeta_\sigma)/\rho_\theta$ is not, in general, materially conserved in adiabatic and frictionless flows along isentropes at the surface.

The quantity $(f + \zeta_\sigma)/\rho_\theta$ is not materially conserved because baroclinicity at the surface can affect the relative vorticity ζ_σ of the surface flow. The contribution $\boldsymbol{\nabla}\cdot(\rho \mathbf{K}_b)$ of the baroclinic component (16b) to the divergence of the surface potential vorticity flux density is equal, up to factors that are constant along isentropes at the surface, to the downward normal component

$$-\mathbf{n} \cdot \boldsymbol{\nabla} \times (c_p T\, \boldsymbol{\nabla} \log \theta) = \boldsymbol{\nabla}\cdot \left(\frac{c_p T}{\theta}\, \mathbf{n} \times \boldsymbol{\nabla}\theta \right)$$

of the baroclinicity vector $-\boldsymbol{\nabla} \times \rho^{-1}\, \boldsymbol{\nabla} p = \boldsymbol{\nabla} \times c_p T\, \boldsymbol{\nabla} \log \theta$. The component of the baroclinicity vector normal to the surface does not generally vanish but affects the surface potential vorticity S via the relative vorticity ζ_σ of the surface flow, giving rise to the baroclinic component of the flux of surface potential vorticity along isentropes at the surface. In contrast, the component of the baroclinicity vector normal to isentropes vanishes, so the isentropic relative vorticity ζ_θ is not affected by baroclinicity and the flux of interior potential vorticity is purely advective in adiabatic and frictionless flows, including flows along isentropes at or immediately above the surface.

The baroclinic component of the surface potential vorticity flux, then, is due to the difference between the relative vorticities ζ_θ and ζ_σ. The baroclinic component $\mathbf{K}_b$ of the surface potential vorticity flux represents the effects of baroclinicity on the relative vorticity ζ_σ of the surface flow.

b. Scale analysis for small Rossby number

For hydrostatic flows with small Rossby number, the ratio of the difference $\zeta_\theta - \zeta_\sigma$ between the relative vorticities to the relative vorticities ζ_θ and ζ_σ themselves scales like the ratio $\mathrm{Fr}^2/\mathrm{Ro}$ of squared Froude number $\mathrm{Fr} = U/(NH)$ to Rossby number $\mathrm{Ro} = U/(fL)$, where U is a velocity scale, N the Brunt-Väisälä frequency, H a height scale, and L a length scale. Since the baroclinic component $\mathbf{K}_b$ of the surface potential vorticity flux is due to the difference $\zeta_\theta - \zeta_\sigma$ between the relative vorticities, the baroclinic component $\mathbf{K}_b$ of the surface potential vorticity flux is of order $O(\mathrm{Fr}^2/\mathrm{Ro})$ compared with the advective component $(u,\ v,\ Q)_\theta S$. That is, for flows with small Rossby number, only if $\mathrm{Fr}^2 \gtrsim \mathrm{Ro}$ can surface baroclinicity lead to significant deviations from material conservation of generalized potential vorticity.

For quasigeostrophic flows on the scale of the Rossby radius $L_{\mathrm{Ro}} = NH/f$, the Froude number Fr is of the same order as the Rossby number Ro, and the baroclinic component $\mathbf{K}_b$ of the surface potential vorticity flux is of order $O(\mathrm{Ro})$ compared with the advective component $(u,\ v,\ Q)_\theta S$. The baroclinic component of the surface potential vorticity flux hence is negligible in quasigeostrophic scaling.[7]

As shown above (section 3f), the baroclinic component of the surface potential vorticity flux vanishes for Boussinesq flows over a flat surface, irrespective of quasigeostrophic scaling. In quasigeostrophic scaling, the baroclinic component of the surface potential vorticity flux is negligible, irrespective of topography. Hence, in large-scale atmospheric flows, the baroclinic component of the surface potential vorticity flux will often be negligible.

c. Example: Wake formation in flows past a mountain

One example of a flow in which non-geostrophic effects and the baroclinic component of the surface potential vorticity flux can play a significant role is stratified flow past a mountain. Smolarkiewicz and Rotunno (1989) have demonstrated with numerical simulations that, in stratified flow that is irrotational upstream of an isolated mountain, a wake with a pair of lee vortices can form downstream of the mountain even when the boundary condition at the mountain is a free-slip condition. A free-slip condition at the surface implies that there cannot be a frictional boundary layer from which vorticity could be transferred into the wake in the interior of the flow. In place of frictional processes, baroclinic effects have been linked to the induction of wake vorticity in flows past a mountain without a frictional boundary layer (Smolarkiewicz and Rotunno 1989; Rotunno et al. 1999). But even in the presence of baroclinicity, interior potential vorticity is materially conserved in adiabatic and frictionless flows, and so baroclinic effects alone cannot be responsible for the induction of potential vorticity in a wake. The interior potential vorticity would have to remain zero throughout adiabatic and frictionless flows past a mountain if it is zero upstream of the mountain. Yet, in simulations with weak frictional dis-

[7] Strictly speaking, it is the divergence $\boldsymbol{\nabla}\cdot(\rho \mathbf{K_b}) = \partial_x(\rho_\theta K_b^x)|_\theta + \partial_y(\rho_\theta K_b^y)|_\theta$ that is negligible in quasigeostrophic scaling. The baroclinic component $\mathbf{K}_b$ may have a non-divergent component that is irrelevant for the transfer of surface potential vorticity.

Quantity	Scale
t	L/U
x, y, z	$H = U/N$
θ'	$\Theta = \theta_0/g\,N^2 H$
P, S	$N\Theta/(\rho_0 L)$
$\mathbf{J}, \mathbf{K}$	$UN\Theta/(\rho_0 L)$
$\boldsymbol{\nabla}\cdot(\rho_0 \mathbf{K})$	$UN\Theta/L^2$

TABLE 1: Scales used for non-dimensionalization of quantities in Figs. 1 and 2. The fundamental scales used for the non-dimensionalization are the horizontal scale L of the mountain and the velocity U and Brunt-Väisälä frequency N of the flow far upstream of the mountain.

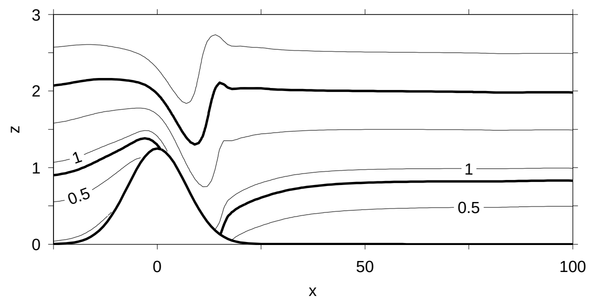

FIGURE 2: Isentropes in the $y = 0$ symmetry plane at time $t = 14.4$. The thin lines represent isentropes ($\theta' = \text{const}$) with a contour interval of $\Delta\theta' = 0.5$. The thick lines represent the surface and the isentropes $\theta' = 0.8$ and $\theta' = 2.0$ on which the generalized potential vorticity is shown in Fig. 1. Potential temperature fluctuations θ' are given in units of the scale $\Theta = \theta_0/g\,N^2 H$ (cf. Tab. 1), so that a unit potential temperature fluctuation $\theta' = 1$ corresponds to a downward displacement of an isentrope by one height scale H.

sipation and diabatic heating, a flow with zero interior potential vorticity upstream of a mountain without a frictional boundary layer can develop a wake with nonzero interior potential vorticity downstream of the mountain (see, e.g., Schär and Durran 1997; Rotunno et al. 1999). The nonzero interior potential vorticity in the wake implies that dissipative processes, however weak, must be active somewhere in the flow. The extent to which baroclinic effects and dissipative processes play a role in the formation of a wake at a mountain without a frictional boundary layer has been the subject of controversy (see, e.g., Smith 1989; Schär and Durran 1997; Rotunno et al. 1999).

The generalized potential vorticity concept allows for a scenario of how a wake can form in weakly dissipative flows past a mountain. Figure 1 shows the time evolution of the surface potential vorticity S and of the interior potential vorticity P on two isentropes during the spin-up of a mountain wake from a potential-flow initial condition. The figure is based on a simulation by Rotunno et al. (1999) of a Boussinesq flow past an isolated mountain. In the simulation, the planetary vorticity is zero, and the Brunt-Väisälä frequency N, the velocity $\mathbf{u} = (U, 0, 0)$, and the surface potential temperature θ_s are uniform far upstream of the mountain, so that the interior potential vorticity P, the surface potential vorticity S, and the generalized potential vorticity $P_g = P + S$ are zero there. The flow impinges along the x-axis upon a radially symmetric Gaussian mountain at the coordinate origin. The mountain is of height $h_M = 1.25H$, where $H = U/N$ is a height scale of the upstream flow, and its slopes are gentle in that the horizontal scale $L = 10H$ of the mountain is considerably greater than the mountain height h_M. The boundary condition at the mountain is a free-slip condition. The only dissipative processes in the simulation are viscous momentum dissipation and thermal diffusion with constant viscosity ν_e and constant thermal diffusivity $\kappa_e = \nu_e$. The Reynolds number $UL/\nu_e = 500$ is chosen small enough that the simulated flow remains laminar, yet large enough that the simulated flow is only weakly dissipative [see Rotunno et al. (1999, section 4) for a detailed description of the simulation].

In the simulation, the formation of a wake with nonzero interior potential vorticity is the result of four processes: *(i)* modification of the thermal stratification in the vicinity of the mountain by gravity waves; *(ii)* baroclinic induction of a surface potential vorticity dipole on the leeward slope of the mountain; *(iii)* downslope advection of the surface potential vorticity dipole and accumulation of surface potential vorticity in a region of large along-stream gradients in surface potential temperature; and *(iv)* dissipative separation of the surface potential vorticity dipole from the surface and advection of an interior potential vorticity dipole along isentropes that intersect the surface.

(i) The way in which gravity waves modify the thermal stratification in the vicinity of the mountain is qualitatively well described by inviscid linear theory. For a mountain that is sufficiently high ($h_M \gtrsim H$), such as the mountain in the simulation considered here ($h_M = 1.25H$), inviscid linear theory predicts that, in the lee of the mountain, isentropes are deflected downward such that they collapse onto the mountain surface (Smith 1988). On the leeward slope of the mountain, therefore, a region of high and relatively uniform surface potential temperature forms, bounded laterally and downslope by regions of large gradients in surface potential

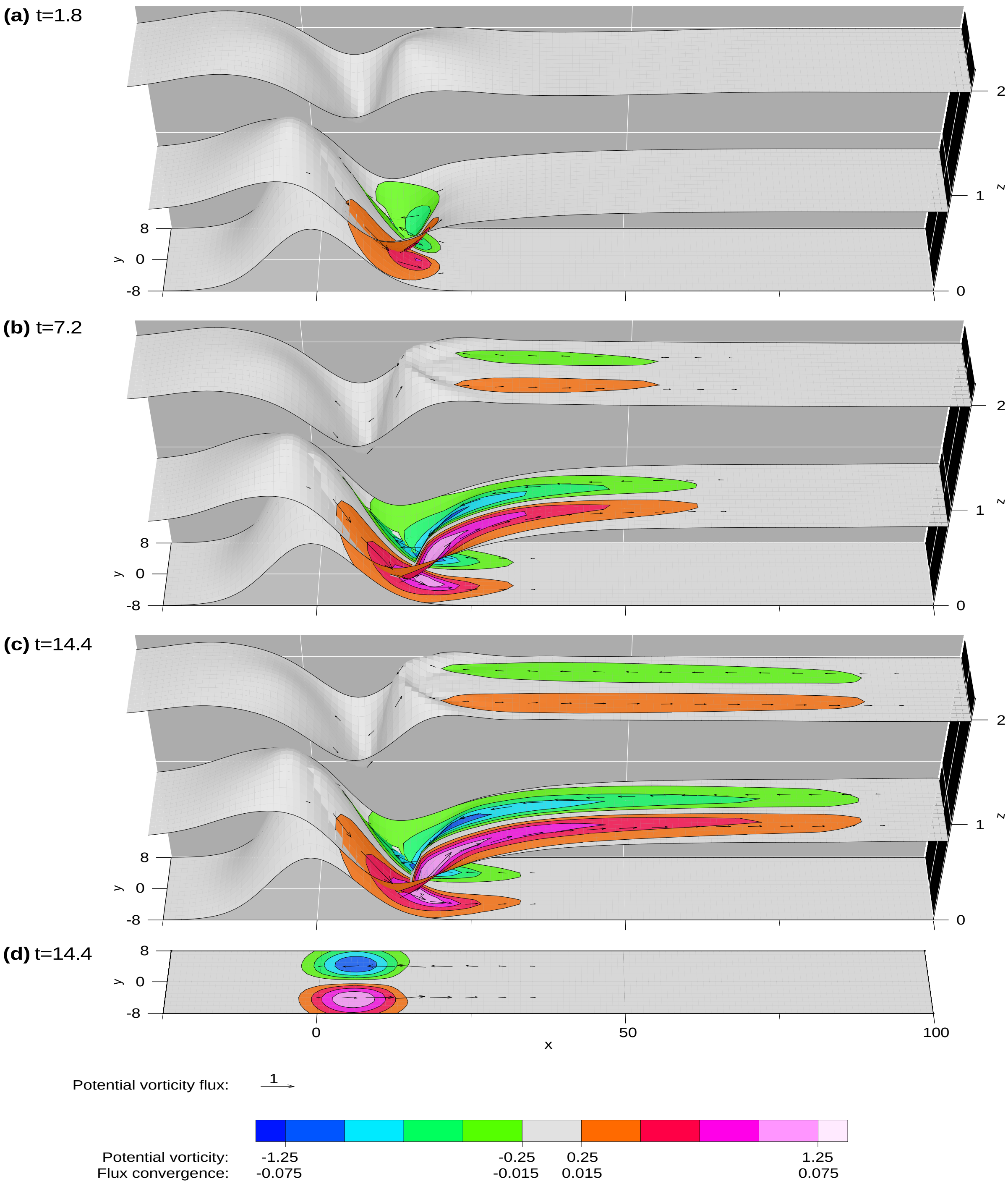

FIGURE 1: Generalized potential vorticity in simulated Boussinesq flow past a mountain. The flow impinges along the x-axis (from the left) upon a radially symmetric mountain at the coordinate origin. Colored contours in panels (a), (b), and (c) indicate the interior potential vorticity (20) on the isentropes $\theta' = 0.8$ and $\theta' = 2.0$ and the surface potential vorticity (21) on the mountain surface for three different times t after the start of the simulation from a potential-flow initial condition. Colored contours in panel (d) indicate, projected onto the xy-plane, the convergence $-\boldsymbol{\nabla}\cdot(\rho_0 \mathbf{K}_b)$ of the baroclinic component (22) of the surface potential vorticity flux. Vectors indicate the magnitude and direction of the advective interior potential vorticity flux $(u, v, 0)_\theta P$ along the isentropes and of the advective surface potential vorticity flux $\mathbf{u}S$. Quantities are given in units of the scales listed in Tab. 1. The delta-function $\delta(z - z_s)$ in the surface potential vorticity was replaced by the inverse height scale $1/H$, so that the surface potential vorticity is finite and of magnitude comparable with the interior potential vorticity.

temperature. Panels (a), (b), and (c) of Fig. 1 each show one of the isentropes ($\theta' = 0.8$) that have collapsed onto the leeward slope of the mountain and one isentrope ($\theta' = 2.0$) that does not intersect the mountain surface (see Tab. 1 for the scales used for nondimensionalization of the potential temperature θ' and other quantities). The essential characteristics of the thermal stratification in the vicinity of the mountain are established at time $t = 1.8$ (Fig. 1a) and evolve only slightly during the further spin-up of the mountain wake (Fig. 1b, c). Figure 2 shows isentropes in the $y = 0$ symmetry plane of the flow at time $t = 14.4$, a time at which a nearly steady state has been reached in the simulation. A region of collapsed isentropes with high and relatively uniform surface potential temperature on the leeward slope and a region of large along-stream gradients in surface potential temperature near the leeward foot of the mountain, as predicted by inviscid linear theory, are clearly recognizable.

(ii) The thermal stratification predicted by inviscid linear theory implies that, in the region of collapsed isentropes on the leeward slope of the mountain, a surface potential vorticity dipole is induced baroclinically. Since the baroclinic component (22) of the surface potential vorticity flux is quadratic in potential temperature fluctuations θ', the baroclinic induction of surface potential vorticity is not taken into account in linear theories [cf. the analysis of Smolarkiewicz and Rotunno (1989), which shows that the baroclinic induction of wake vorticity is an effect of second order in perturbation amplitude]. The across-stream component

$$K_b^y = \frac{\mu}{\rho_0} \frac{g{\theta'}^2}{2\theta_0} (\partial_x z_s)\, \delta(z - z_s)$$

dominates the baroclinic component of the surface potential vorticity flux in the region of collapsed isentropes and transfers surface potential vorticity from the left (facing downstream) of the $y = 0$ symmetry plane to the right (since $K_b^y < 0$ on the leeward slope). Figure 1d shows, at time $t = 14.4$ and projected onto the xy-plane, the convergence $-\boldsymbol{\nabla}\cdot(\rho_0 \mathbf{K}_b)$ of the baroclinic component of the surface potential vorticity flux. Since the baroclinic component of the surface potential vorticity flux depends only on the topography and on the surface potential temperature field θ', which is already close to its steady state at time $t = 1.8$, the structure of the convergence $-\boldsymbol{\nabla}\cdot(\rho_0 \mathbf{K}_b)$ at the earlier times $t = 1.8$ and 7.2 is similar to the shown convergence at time $t = 14.4$. The convergence of the diabatic and frictional components of the generalized potential vorticity flux is, in the region of collapsed isentropes, an order of magnitude smaller than the convergence of the baroclinic component.[8] The convergence of the baroclinic component of the surface potential vorticity flux on the leeward slope of the mountain leads to a surface potential vorticity dipole with negative surface potential vorticity to the left of the $y = 0$ symmetry plane and positive surface potential vorticity to the right.

(iii) The baroclinically induced surface potential vorticity dipole is advected downslope through the region of collapsed isentropes into the region of large along-stream gradients in surface potential temperature near the leeward foot of the mountain. Since a flow with weak thermal diffusion as the only diabatic process effectively cannot cross isentropes at the surface, the advective surface potential vorticity flux converges and surface potential vorticity accumulates in the region of large surface potential temperature gradients. In Fig. 1, the convergence of the advective surface potential vorticity flux can be inferred from the vectors along the surface, which indicate the magnitude and direction of the advective surface potential vorticity flux. The accumulation of surface potential vorticity in the region of large surface potential temperature gradients near the leeward foot of the mountain is clearly recognizable in the succession of panels (a), (b), and (c).

(iv) As surface potential vorticity accumulates in the region of large surface potential temperature gradients near the leeward foot of the mountain, the magnitude of the conversion rate (19) between surface potential vorticity and interior potential vorticity increases on both sides of the $y = 0$ symmetry plane. Eventually, even in weakly dissipative flows, the increasing conversion of surface potential vorticity into interior potential vorticity results in the surface potential vorticity dipole being separated dissipatively from the surface and being advected with the flow as an interior potential vorticity dipole. In a steady state, a balance is established

[8] Scale analysis gives an indication of the relative magnitudes of the baroclinic and dissipative contributions to the convergence of the generalized potential vorticity flux. Integrating the convergence $-\boldsymbol{\nabla}\cdot(\rho_0 \mathbf{J}_g)$ of the generalized potential vorticity flux over a volume V that encloses a surface patch of area A and that has infinitesimally small faces normal to the surface, one finds that, at about half-height on the leeward slope, the baroclinic contribution $-\int_V \boldsymbol{\nabla}\cdot(\rho_0 \mathbf{K}_b)\, d\mathbf{x}$ to the integral scales like $\frac{g}{\theta_0} \frac{\Theta^2 h_M}{L^2} A$, whereas the dissipative contributions $-\int_V \boldsymbol{\nabla}\cdot(\rho_0 \mathbf{K}_F)\, d\mathbf{x} - \int_A \mathbf{n}\cdot(\rho_0 \mathbf{J})\, dA = \int_A \big[Q\zeta_\sigma + \theta' \mathbf{n}\cdot(\boldsymbol{\nabla}\times \mathbf{F})\big]\, dA$ scale like $\frac{\nu_e \Theta U}{H^2 L} A$ (cf. the scales in Tab. 1). The ratio of the baroclinic contribution to the dissipative contributions is therefore $\frac{N^2 H^3 h_M}{\nu_e U L} = \frac{UL}{\nu_e} \cdot \frac{H h_M}{L^2} \approx 6$ (since $UL/\nu_e = 500$ and $L = 8h_M = 10H$ in the simulation considered here). This scaling estimate underestimates the actual ratio of the baroclinic contribution to the dissipative contributions — largely, it appears, because the scaling estimate $\zeta_\sigma \sim U/L$ for the vorticity in the dissipative term overestimates the vorticity in the region on the leeward slope where the baroclinic contribution to the convergence of the generalized potential vorticity flux is largest.

between the convergence of surface potential vorticity flux and the conversion of surface potential vorticity into interior potential vorticity. This balance is a consequence of the fact that, in a steady state, the divergence $\nabla\cdot(\rho_0 \mathbf{J}_g) = \nabla\cdot(\rho_0\mathbf{J} + \rho_0\mathbf{K})$ of the generalized potential vorticity flux vanishes, whence, by integration over a volume V that encloses a surface patch of area A and that has infinitesimally small faces normal to the surface, it follows that

$$-\int_V \nabla\cdot(\rho_0\mathbf{K})\, d\mathbf{x} = \int_A \mathbf{n}\cdot(\rho_0\mathbf{J})\, dA. \qquad (31)$$

In the region of large surface potential temperature gradients near the leeward foot of the mountain, the baroclinic component $\mathbf{K}_b \propto \theta'^2(\mathbf{n}\times\mathbf{k})$ is negligible in the convergence of the surface potential vorticity flux (cf. Fig. 1d) because the normal vector $\mathbf{n}$ is nearly parallel to the vertical unit vector $\mathbf{k}$. The balance (31) between the convergence of surface potential vorticity flux and the conversion of surface potential vorticity into interior potential vorticity can therefore be approximated as

$$-\int_V \nabla\cdot(\rho_0\mathbf{u}S)\, d\mathbf{x} \approx -\int_A \Big[Q\zeta_\sigma + \theta'\mathbf{n}\cdot\big(\nabla\times\mathbf{F}\big)\Big]\, dA,$$

where we have combined the frictional components $\mathbf{K}_F$ and $\mathbf{J}_F$ of the surface potential vorticity flux and of the interior potential vorticity flux in the surface integral on the right hand side (cf. footnotes 3 and 4). Since the region of large surface potential temperature gradients near the leeward foot of the mountain is anomalously warm (cf. Fig. 2), the diffusive diabatic heating rate Q tends to be negative there, so that, since the surface potential temperature anomaly θ' is positive on the leeward slope of the mountain, the diabatic conversion $-Q\zeta_\sigma$ damps surface potential vorticity anomalies $S \propto \theta'\zeta_\sigma$ by converting positive surface potential vorticity into positive interior potential vorticity and negative surface potential vorticity into negative interior potential vorticity. The magnitude of the conversion rate $-Q\zeta_\sigma$ increases with increasing magnitude of the surface potential vorticity anomaly $S \propto \theta'\zeta_\sigma$. The frictional conversion $-\theta'\mathbf{n}\cdot(\nabla\times\mathbf{F})$ has a similar effect: since extrema in the relative vorticity $\zeta_\sigma = \mathbf{n}\cdot\boldsymbol{\omega}_\mathbf{r}$ are concomitant with the extrema in surface potential vorticity, the frictional conversion rate $-\theta'\mathbf{n}\cdot(\nabla\times\mathbf{F}) = -\nu_e\theta'\mathbf{n}\cdot(\nabla^2\boldsymbol{\omega}_\mathbf{r})$ is positive (negative) at the surface potential vorticity maximum (minimum) near the leeward foot of the mountain.

Dissipative conversion of surface potential vorticity into interior potential vorticity thus counteracts the accumulation of surface potential vorticity due to the convergence of the advective surface potential vorticity flux and implies the separation of the surface potential vorticity dipole from the surface. Separated from the surface, the surface potential vorticity dipole is advected with the flow as an interior potential vorticity dipole (Fig. 1). Since the flow is only weakly dissipative, the interior potential vorticity dipole originating in the surface potential vorticity sheet is advected predominantly along isentropes — along isentropes that intersect the surface near the leeward foot of the mountain. On isentropes that do not intersect the surface, such as the upper isentropes in Fig. 1, interior potential vorticity cannot be induced by conversion from surface potential vorticity, but arises by vertical diffusion of interior potential vorticity from lower-lying isentropes that intersect the surface. The interior potential vorticity anomalies are therefore weaker on isentropes that do not intersect the surface than on isentropes that do intersect the surface (Fig. 1).

In this scenario, a wake with a pair of counterrotating lee vortices forms by conversion of a baroclinically induced surface potential vorticity dipole into an interior potential vorticity dipole. As in the analysis of Rotunno et al. (1999), baroclinicity is posited as fundamental for the formation of a wake with nonzero interior potential vorticity. The generalized potential vorticity perspective emphasizes the baroclinicity near the surface of the mountain and shows that the principal role of dissipation is to separate from the surface the surface potential vorticity dipole that develops on the leeward slope of the mountain and is advected downslope. The description from the perspective of generalized potential vorticity resembles the classical descriptions of the separation of a vorticity sheet induced by frictional processes at a boundary. The main difference from the case of a friction-induced vorticity sheet at a surface with a no-slip boundary condition is that in the case of a free-slip boundary condition, a potential vorticity sheet can be induced by baroclinicity at the boundary.

6. Mean budget of generalized potential vorticity in isentropic coordinates

Replacing the inhomogeneous boundary condition at the surface by a homogeneous boundary condition plus boundary sources, we have formally extended the entropic flow domain to all potential temperatures $\theta > \theta_b = 0$ and have thus made it time-independent. In the extended time-independent domain, the generalized potential vorticity and the generalized potential vorticity flux with the included boundary sources replace the potential vorticity and the potential vorticity flux of the original time-dependent domain. Consistent with the homogeneous boundary conditions $\mathbf{D}_b = 0$ and $\mathbf{H}_b = 0$

for the fields $\mathbf{D}$ and $\mathbf{H}$ inside the surface, the contributions of the interior potential vorticity (25) and of its flux (27) to the generalized potential vorticity and its flux are set to zero on isentropes with potential temperature $\theta < \theta_s(x, y, t)$ less than the surface potential temperature $\theta_s(x, y, t)$. On isentropes $\theta = \theta_s(x, y, t)$ that intersect the surface, the surface potential vorticity (26) and its flux (28) contribute as singular boundary sources to the generalized potential vorticity and its flux.

With the generalized potential vorticity and its flux defined in a time-independent entropic domain, one can define their isentropic mean fields throughout the entire entropic flow domain, including the surface layer of isentropes that sometimes intersect the surface. In the surface layer, both the interior potential vorticity and the surface potential vorticity contribute to the mean budget of generalized potential vorticity; in the interior atmosphere above the surface layer, only the interior potential vorticity contributes to the mean budget of generalized potential vorticity. In what follows, we will discuss the structure of the mean budget of generalized potential vorticity for flows with stationary and axisymmetric flow statistics, for which mean fields are functions of latitude y and potential temperature θ.

a. Definition of mean fields and eddy fields

For the discussion of the mean of the singular surface contributions to the generalized potential vorticity budget, it is convenient to define mean fields explicitly as probabilistic means and eddy fields as fluctuations about these means. Since we are considering compressible flows, the mean field of a scalar field $A(x, y, \theta, t)$ is defined as the density-weighted mean (cf. Gallimore and Johnson 1981; Tung 1986)

$$\overline{A}^*(y, \theta) = \frac{\overline{\rho_\theta A}}{\bar{\rho}_\theta}. \qquad (32)$$

The overbar $\bar{B}$ on a scalar field $B = B\big(\boldsymbol{\gamma}(x, y, \theta, t)\big)$ that is a local function of some or all components of the state vector $\boldsymbol{\gamma}(x, y, \theta, t)$ characterizing the flow at position (x, y, θ) and time t denotes the probabilistic mean (cf. Monin and Yaglom 1971, section 3)

$$\bar{B}(y, \theta) = \int B(\boldsymbol{\gamma}')\, \pi^{\boldsymbol{\gamma}}_{y,\theta}(\boldsymbol{\gamma}')\, d\boldsymbol{\gamma}'.$$

The probabilistic mean is the expectation value with respect to the probability density $\pi^{\boldsymbol{\gamma}}_{y,\theta}(\boldsymbol{\gamma}')$ of the state vector $\boldsymbol{\gamma}$, where the probability density $\pi^{\boldsymbol{\gamma}}_{y,\theta}(\boldsymbol{\gamma}')$ depends parametrically on latitude y and potential temperature θ.[9] Eddy fields are defined as fluctuations

$$\hat{A}(x, y, \theta, t) = A\big(\boldsymbol{\gamma}(x, y, \theta, t)\big) - \overline{A}^*(y, \theta) \qquad (33)$$

about the density-weighted mean $\overline{A}^*$ (cf. Gallimore and Johnson 1981; Tung 1986).

The isentropic density $\rho_\theta(x, y, \theta, t)$ appearing in the density-weighted mean (32) must be set to zero on isentropes $\theta < \theta_s(x, y, t)$ inside the surface for the vertical integral of the isentropic density $\rho_\theta(x, y, \theta, t)$ over all potential temperatures $\theta > \theta_b$ to be equal to the atmospheric mass per unit area. So the mean isentropic density is the ensemble mean

$$\bar{\rho}_\theta(y, \theta) = \overline{\rho_\theta\, \mathcal{H}(\theta - \theta_s)} \qquad (34)$$

of the product of isentropic density ρ_θ and step function $\mathcal{H}(\theta - \theta_s)$.[10] Since, by assumption, the isentropic density $\rho_\theta(x, y, \theta, t)$ is greater than zero on isentropes $\theta > \theta_s(x, y, t)$ above the surface, such that isentropic coordinates are well-defined, the mean isentropic density $\bar{\rho}_\theta(y, \theta)$ at a given latitude y is greater than zero on all isentropes θ with nonzero probability of being above the surface. The mean isentropic density $\bar{\rho}_\theta(y, \theta)$ at a given latitude y is zero on isentropes θ with vanishing probability of being above the surface. Since the mean isentropic density $\bar{\rho}_\theta(y, \theta)$ appears in the denominator of the density-weighted mean (32), density-weighted mean fields (32) and eddy fields (33) at a given latitude y are defined only on isentropes θ with nonzero probability of being above the surface.

b. Mean surface potential vorticity

The surface potential vorticity $S(x, y, \theta, t)$ is nonzero only on isentropes $\theta = \theta_s(x, y, t)$ at the surface. Instantaneous surface potential vorticities $S(x, y, \theta, t)$ hence contribute to the mean surface potential vorticity

$$\overline{S}^*(y, \theta) = \theta\, \frac{\overline{(f + \zeta_\sigma)\, \delta(\theta - \theta_s)}}{\bar{\rho}_\theta} \qquad (35)$$

at a given latitude y only on isentropes θ with nonzero probability of intersecting the surface.

[9] We use the symbol $\pi^{\psi}_{\mathbf{x}}(\cdot)$ as the generic symbol for a probability density. The superscript ψ indicates the random variable to which the probability density $\pi^{\psi}_{\mathbf{x}}(\cdot)$ belongs; the subscript $\mathbf{x}$ indicates the coordinates on which the probability density depends; and the argument $(\cdot)$ indicates the realization at which the probability density is evaluated.

[10] If there is a mixed layer with vertically uniform potential temperature $\theta = \theta_s$ adjacent to the surface, the mean isentropic density becomes the sum $\bar{\rho}_\theta(y, \theta) = \overline{\rho_\theta\, \mathcal{H}(\theta - \theta_s)} + \overline{\sigma_m\, \delta(\theta - \theta_s)}$ of the mean interior density $\overline{\rho_\theta\, \mathcal{H}(\theta - \theta_s)}$ and a contribution $\overline{\sigma_m\, \delta(\theta - \theta_s)}$ from the mixed layer, where σ_m is the mixed layer mass per unit area (cf. Held and Schneider 1999). In the present paper, we disregard the additional complications that a mixed layer poses for defining isentropic mean fields.

If the relative vorticity ζ_σ of the surface flow is negligible compared with the planetary vorticity f, the mean surface potential vorticity can be approximated by

$$\overline{S}^*(y,\theta) \approx \frac{f}{\bar{\rho}_\theta}\,\theta\,\overline{\delta(\theta-\theta_s)}.$$

The mean of the delta-function,[11]

$$\overline{\delta(\theta-\theta_s)} = \pi_y^{\theta_s}(\theta),$$

is the probability density $\pi_y^{\theta_s}(\theta)$ of the surface potential temperature, so that the mean surface potential vorticity is approximately

$$\overline{S}^*(y,\theta) \approx \frac{f}{\bar{\rho}_\theta}\,\theta\,\pi_y^{\theta_s}(\theta).$$

In this approximation, it is evident that, if the mean isentropic density $\bar{\rho}_\theta(y,\theta)$ is a smooth function of potential temperature θ, the mean surface potential vorticity $\overline{S}^*(y,\theta)$ varies as smoothly with potential temperature θ as the probability density $\pi_y^{\theta_s}(\theta)$ of the surface potential temperature. Being distributed over the range of surface potential temperatures — the range of potential temperatures θ for which the probability density $\pi_y^{\theta_s}(\theta)$ is nonzero — the singular contributions of the instantaneous surface potential vorticities $S(x,y,\theta,t)$ are smoothed out in the isentropic mean $\overline{S}^*(y,\theta)$.

That region of the entropic flow domain in which potential temperatures lie within the range of surface potential temperatures, such that the mean surface potential vorticity $\overline{S}^*(y,\theta)$ is nonzero, will be referred to as the surface layer. That region of the entropic flow domain in which potential temperatures lie above the range of surface potential temperatures, such that the mean surface potential vorticity $\overline{S}^*(y,\theta)$ is zero, will be referred to as the interior atmosphere (cf. Held and Schneider 1999).

[11] The mean

$$\overline{\delta(\theta-\theta_s)} = \int \delta(\theta-\theta_s')\,\pi_{y,\theta}^{\boldsymbol{\gamma}}(\boldsymbol{\gamma}')\,d\boldsymbol{\gamma}'$$

of the delta-function can be evaluated as follows. Decomposing the state vector $\boldsymbol{\gamma} = (\theta_s, \boldsymbol{\gamma}_-)$ into the surface potential temperature θ_s as one component and a vector $\boldsymbol{\gamma}_-$ consisting of all other components of the state vector, we can write the mean as

$$\overline{\delta(\theta-\theta_s)} = \int \delta(\theta-\theta_s')\left(\int \pi_{y,\theta}^{\boldsymbol{\gamma}}(\theta_s',\boldsymbol{\gamma}_-')\,d\boldsymbol{\gamma}_-'\right)d\theta_s',$$

because the delta-function $\delta(\theta-\theta_s)$ depends only on the surface potential temperature θ_s and not on other components of the state vector $\boldsymbol{\gamma}$. The inner integral, which extends over all components $\boldsymbol{\gamma}_-$ of the state vector except the surface potential temperature, yields the probability density

$$\pi_y^{\theta_s}(\theta_s') = \int \pi_{y,\theta}^{\boldsymbol{\gamma}}(\theta_s',\boldsymbol{\gamma}_-')\,d\boldsymbol{\gamma}_-'$$

of the surface potential temperature (cf. Papoulis 1991, chapter 7.3). Carrying out the remaining integration $\int \delta(\theta-\theta_s')\,\pi_y^{\theta_s}(\theta_s')\,d\theta_s'$ gives $\overline{\delta(\theta-\theta_s)} = \pi_y^{\theta_s}(\theta)$.

c. Mean interior potential vorticity

The interior potential vorticity $P(x,y,\theta,t)$ is generally nonzero on all isentropes $\theta > \theta_s(x,y,t)$ above the surface. Instantaneous interior potential vorticities $P(x,y,\theta,t)$ hence contribute to the mean interior potential vorticity

$$\overline{P}^*(y,\theta) = \frac{\overline{(f+\zeta_\theta)\,\mathcal{H}(\theta-\theta_s)}}{\bar{\rho}_\theta} \tag{36}$$

at a given latitude y on all isentropes θ with nonzero probability of being above the surface. That is, the mean interior potential vorticity $\overline{P}^*(y,\theta)$ is generally nonzero both in the interior atmosphere and in the surface layer.

In the interior atmosphere, the mean interior potential vorticity is the conventional density-weighted mean $\overline{P}^*(y,\theta) = \overline{\rho_\theta P}/\bar{\rho}_\theta = \overline{(f+\zeta_\theta)}/\bar{\rho}_\theta$ (cf. Tung 1986). In the surface layer, the mean interior potential vorticity can be estimated similarly to how the mean surface potential vorticity was estimated if the relative vorticity ζ_θ is negligible compared with the planetary vorticity f. The mean interior potential vorticity (36) can then be approximated by

$$\overline{P}^*(y,\theta) \approx \frac{f}{\bar{\rho}_\theta}\,\overline{\mathcal{H}(\theta-\theta_s)}.$$

Evaluating the ensemble mean of the step function similarly to the ensemble mean of the delta-function (cf. footnote 11), the ensemble mean of the step function

$$\overline{\mathcal{H}(\theta-\theta_s)} = \Pi_y^{\theta_s}(\theta)$$

yields the cumulative distribution

$$\Pi_y^{\theta_s}(\theta) = \int_{\theta_b}^{\theta} \pi_y^{\theta_s}(\theta_s')\,d\theta_s'$$

of surface potential temperatures. [The value $\Pi_y^{\theta_s}(\theta)$ of the cumulative distribution function indicates the probability that the surface potential temperature $\theta_s(x,y,t)$ at latitude y is less than a given value θ, or equivalently, the probability that the isentrope with potential temperature θ is above the surface.] The mean interior potential vorticity is therefore approximately

$$\overline{P}^*(y,\theta) \approx \frac{f}{\bar{\rho}_\theta}\,\Pi_y^{\theta_s}(\theta).$$

If one makes the additional assumption of a nearly constant isentropic density $\rho_\theta(x,y,\theta,t) \approx \rho_\theta^0(y)$ near the surface — corresponding to the quasigeostrophic assumption of constant static stability, except that here the isentropic density $\rho_\theta^0(y)$ can be a function of latitude y — the mean isentropic density (34) becomes

$$\bar{\rho}_\theta(y,\theta) \approx \rho_\theta^0(y)\,\Pi_y^{\theta_s}(\theta).$$

With this approximation for the mean isentropic density, the mean interior potential vorticity simplifies to

$$\overline{P}^*(y,\theta) \approx \frac{f}{\rho_\theta^0}, \tag{37}$$

a function only of latitude y. In Section 6e, in deriving a relationship between eddy fluxes and the mean meridional mass flux along isentropes, we will use the approximation that the mean interior potential vorticity in the surface layer is a function only of latitude. Since this approximation presumes that the relative vorticity ζ_θ is negligible compared with the planetary vorticity f, it can be expected to be adequate in the extratropics, but not necessarily in the tropics.

d. Conservation of mean generalized potential vorticity

The preceding discussion of the mean interior potential vorticity and of the mean surface potential vorticity implies that the mean generalized potential vorticity $\overline{P_g}^* = \overline{P}^* + \overline{S}^*$ typically makes a smooth transition from the interior atmosphere, in which the mean generalized potential vorticity is equal to the mean interior potential vorticity $\overline{P}^*$, to the surface layer, in which both the mean interior potential vorticity $\overline{P}^*$ and the mean surface potential vorticity $\overline{S}^*$ contribute to the mean generalized potential vorticity. Similarly, the singular contributions of the instantaneous surface potential vorticity flux $\mathbf{K}(x,y,\theta,t)$ will usually be smoothed out in the isentropic mean $\overline{\mathbf{J}_g}^* = \overline{\mathbf{J}}^* + \overline{\mathbf{K}}^*$ of the generalized potential vorticity flux, with the mean interior potential vorticity flux $\overline{\mathbf{J}}^*$ contributing both in the interior atmosphere and in the surface layer, and with the mean surface potential vorticity flux $\overline{\mathbf{K}}^*$ contributing only in the surface layer. Therefore, the terms in the conservation law

$$\partial_t\big(\bar\rho_\theta\,\overline{P_g}^*\big) + \partial_x\big(\bar\rho_\theta\,\overline{J_g^x}^*\big) + \partial_y\big(\bar\rho_\theta\,\overline{J_g^y}^*\big) + \partial_\theta\big(\bar\rho_\theta\,\overline{J_g^\theta}^*\big) = 0 \tag{38}$$

for the mean generalized potential vorticity $\overline{P_g}^*$, obtained by averaging the generalized potential vorticity conservation law (30), will usually vary smoothly throughout the entropic flow domain.

The conservation law (38) is exact within the approximations of the hydrostatic primitive equations and allows one to consider an isentropic mean budget of potential vorticity throughout the entire entropic flow domain, including the surface layer of isentropes that sometimes intersect the surface. The conservation law for the mean generalized potential vorticity might be analyzed, for example, to study the formation of a turbulent wake — as the conservation law for the instantaneous generalized potential vorticity was analyzed in section 5c to study the formation of a laminar wake. Or the conservation law for the mean generalized potential vorticity might be analyzed to study the exchange of potential vorticity between the surface layer and the interior of a flow — for example, as Marshall and Nurser (1992) analyzed the mean potential vorticity budget of thermocline ventilation.

For statistically stationary and axisymmetric flows, the derivatives with respect to longitude x and time t in the conservation law (38) vanish,

$$\partial_y\big(\bar\rho_\theta\,\overline{J_g^y}^*\big) + \partial_\theta\big(\bar\rho_\theta\,\overline{J_g^\theta}^*\big) = 0. \tag{39}$$

From this form of the conservation law for the mean generalized potential vorticity, one can deduce a balance condition that relates the mean meridional mass flux along isentropes to eddy fluxes of interior potential vorticity and of surface potential temperature.

e. Example: Eddy fluxes and the mean meridional mass flux along isentropes

The conservation law (39) for the mean generalized potential vorticity of statistically stationary and axisymmetric flows takes a particularly simple form when the generalized potential vorticity flux $\mathbf{J}_g = \mathbf{J}+\mathbf{K}$ is derived from the gauge given by Eqs. (6) and (10). The surface potential vorticity flux $\mathbf{K} = \rho^{-1}(\mathbf{n}\times\mathbf{H}_s)\,\delta(z-z_s)$ is, in this gauge,

$$\mathbf{K} = -\rho^{-1}\big[B(\mathbf{n}\times\boldsymbol{\nabla}\theta) + \partial_t\theta(\mathbf{n}\times\mathbf{u_a})\big]\,\delta(z-z_s),$$

which in isentropic coordinates becomes

$$\mathbf{K} = \frac{\mu B}{\rho_\theta}\big(\theta_s^y,\ -\theta_s^x,\ 0\big)_\theta\,\delta(\theta-\theta_s). \tag{40}$$

The horizontal derivatives θ_s^x and θ_s^y of the surface potential temperature $\theta_s(x,y,t)$ are given by Eq. (29), and, in the hydrostatic approximation, the Bernoulli function $B = \frac{1}{2}\|\mathbf{v}\|^2 + c_pT + \Phi$ contains only the horizontal velocity $\mathbf{v} = (u,\ v,\ 0)_\theta$. Unlike the surface potential vorticity flux (28) in the gauge given by Eqs. (11) and (13), the surface potential vorticity flux (40) in the gauge given by Eqs. (6) and (10) has no cross-isentropic component. Since the interior potential vorticity flux (27) likewise has no cross-isentropic component, the generalized potential vorticity flux has no cross-isentropic component in this gauge, so that, in the conservation law (39), the derivative with respect to potential temperature θ vanishes. Integrating over latitude y, using the no-flow boundary condition at the poles, and substituting the interior potential vorticity flux (27) and the surface potential vorticity flux (40), one finds the form

$$\bar{\rho}_\theta\, \overline{vP}^* + \bar{\rho}_\theta\, \overline{J_Q^y}^* + \bar{\rho}_\theta\, \overline{J_F^y}^* - \overline{\mu B \theta_s^x\, \delta(\theta - \theta_s)} = 0 \tag{41}$$

for the conservation law (39). In this form, the mean budget of generalized potential vorticity is a statement of mean zonal momentum balance on isentropes. In the interior atmosphere, only the interior potential vorticity flux (the first three terms) is nonzero; the generalized potential vorticity budget reduces to the well-known budget of interior potential vorticity, which is identical with the mean zonal momentum balance (cf. Tung 1986; Andrews et al. 1987, chapter 3). In the surface layer, the surface potential vorticity flux (the fourth term) contributes to the generalized potential vorticity budget.

A relationship between the mean meridional mass flux along isentropes and components of the mean generalized potential vorticity flux follows if the advective interior potential vorticity flux $\overline{vP}^* = \overline{v}^* \overline{P}^* + \overline{\hat{v}\hat{P}}^*$ is decomposed into a mean advective flux $\overline{v}^* \overline{P}^*$ and an eddy flux $\overline{\hat{v}\hat{P}}^*$. Dividing the mean budget of generalized potential vorticity (41) by the mean interior potential vorticity $\overline{P}^*$ and rearranging terms yields

$$\bar{\rho}_\theta\, \overline{v}^* = -\frac{1}{\overline{P}^*} \Big[\bar{\rho}_\theta\, \overline{\hat{v}\hat{P}}^* - \overline{\mu B \theta_s^x\, \delta(\theta - \theta_s)} + \bar{\rho}_\theta\, \overline{J_Q^y}^* + \bar{\rho}_\theta\, \overline{J_F^y}^* \Big]. \tag{42}$$

This alternative form of the mean generalized potential vorticity budget is valid where the mean interior potential vorticity $\overline{P}^*$ is nonzero. In this form, the mean budget of generalized potential vorticity represents a balance condition that relates the mean meridional mass flux $\bar{\rho}_\theta\, \overline{v}^*$ along isentropes to components of the mean fluxes of interior and surface potential vorticity.

How the contribution $\overline{\mu B \theta_s^x\, \delta(\theta - \theta_s)}$ of the surface potential vorticity flux is to be interpreted becomes clearer when the generalized potential vorticity budget (42) is integrated vertically over the surface layer, from the potential temperature θ_b of the lower boundary of the entropic flow domain to the potential temperature θ_i of some isentrope in the interior atmosphere. Using the hydrostatic approximation $\mu \approx 1$ and the approximation (37) that the mean interior potential vorticity $\overline{P}^* \approx f/\rho_\theta^0$ in the surface layer is a function only of latitude, so that it can be taken outside the integral, one obtains for the integrated contribution of the surface potential vorticity flux

$$\begin{aligned}\int_{\theta_b}^{\theta_i} \frac{\overline{\mu B \theta_s^x\, \delta(\theta - \theta_s)}}{\overline{P}^*}\, d\theta &\approx \frac{\rho_\theta^0}{f} \int_{\theta_b}^{\theta_i} \overline{B \theta_s^x\, \delta(\theta - \theta_s)}\, d\theta \\ &= \frac{\rho_\theta^0}{f}\, \overline{B_s \theta_s^x}^s. \end{aligned} \tag{43}$$

Integrated over the surface layer, the isentropic mean $\overline{(\cdot)\, \delta(\theta - \theta_s)}$ becomes a mean $\overline{(\cdot)}^s$ of surface quantities (marked by the subscript s). Since horizontal gradients in kinetic energy $\frac{1}{2}\|\mathbf{v}_s\|^2$ at the surface are typically much smaller than horizontal gradients in enthalpy $c_p T_s$, the surface Bernoulli function B_s in the last line of Eq. (43) can be approximated by the Montgomery streamfunction $M_s = c_p T_s + g z_s$. Using this approximation, integrating by parts,

$$\overline{B_s \theta_s^x}^s = -\overline{(\partial_x B_s)\theta_s}^s \approx -\overline{(\partial_x M_s)\theta_s}^s,$$

and introducing the balanced meridional velocity $\tilde{v}_s$ at the surface by

$$\tilde{v}_s = f^{-1} \partial_x M_s, \tag{44}$$

one finds for the integral (43) the approximation

$$\int_{\theta_b}^{\theta_i} \frac{\overline{\mu B \theta_s^x\, \delta(\theta - \theta_s)}}{\overline{P}^*}\, d\theta \approx -\rho_\theta^0\, \overline{\tilde{v}_s' \theta_s'}^s.$$

The fluctuations $(\cdot)' = (\cdot) - \overline{(\cdot)}^s$ about the surface mean $\overline{(\cdot)}^s$ appear on the right-hand side because the mean $\overline{\tilde{v}_s}^s$ of the balanced meridional velocity vanishes. At any given latitude, the contribution of the surface potential vorticity flux to the mean meridional mass flux (42) integrated over the surface layer is approximately proportional to the balanced eddy flux $\overline{\tilde{v}_s' \theta_s'}^s$ of surface potential temperature, with the near-surface isentropic density ρ_θ^0 as constant of proportionality. In the Boussinesq limit, the balanced meridional velocity $\tilde{v}_s$ is the geostrophic meridional velocity, and the contribution of the surface potential vorticity flux to the integrated mean meridional mass flux is approximately proportional to the geostrophic eddy flux of surface potential temperature.

Integrated vertically over the surface layer, then, the balance condition (42) relating the mean meridional mass flux along isentropes to components of the mean fluxes of interior and surface potential vorticity flux becomes approximately

$$\int_{\theta_b}^{\theta_i} \bar{\rho}_\theta\,\overline{v}^*\,d\theta$$
$$\approx -\int_{\theta_b}^{\theta_i} \frac{\bar{\rho}_\theta\,\overline{\hat{v}\hat{P}}^* + \bar{\rho}_\theta\,\overline{J_F^y}^*}{\overline{P}^*}\,d\theta - \rho_\theta^0\,\overline{\tilde{v}_s'\theta_s'}^s, \quad (45)$$

where we have neglected the diabatic component $\overline{J_Q^y}^*$ of the interior potential vorticity flux, because it is smaller than the eddy flux $\overline{\hat{v}\hat{P}}^*$ by a factor of order Rossby number (Haynes and McIntyre 1987). This approximate balance condition between the integrated mean meridional mass flux along isentropes on the one hand and components of the mean interior potential vorticity flux and the balanced eddy flux of surface potential temperature on the other hand holds in the extratropics, where the Rossby number is small. In the extratropics, the mean generalized potential vorticity budget (41) can be divided by the mean interior potential vorticity $\overline{P}^* \neq 0$, the mean interior potential vorticity in the surface layer can be approximated by $\overline{P}^* \approx f/\rho_\theta^0$, and a balanced meridional velocity $\tilde{v}_s$ at the surface can be defined as being proportional to the along-surface derivative (44) of the Montgomery streamfunction M_s. Using a more restrictive set of assumptions, Held and Schneider (1999) argued that, in the extratropics, the mean meridional mass flux in the surface layer contains a component proportional to the geostrophic eddy flux of surface potential temperature and that this component dominates the mean meridional mass flux in the surface layer. The approximate balance condition (45) shows in more detail how not only the balanced eddy flux $\overline{\tilde{v}_s'\theta_s'}^s$ of surface potential temperature, but also the eddy component $\overline{\hat{v}\hat{P}}^*$ and the frictional component $\overline{J_F^y}^*$ of the interior potential vorticity flux contribute to the mean meridional mass flux in the surface layer.

Figure 3 shows the mass flux streamfunction

$$\Psi(\phi,\theta) = 2\pi a\cos(\phi)\int_{\theta_b}^{\theta} \bar{\rho}_\theta\,\overline{v}^*\,d\theta'$$

in a simulation with an idealized GCM. The mass flux streamfunction Ψ is the integrated mean meridional mass flux multiplied by the lengths $2\pi a\cos(\phi)$ of latitude circles (where a is the radius of the planet and ϕ is latitude). The idealized GCM is a primitive-equation model of an ideal-gas atmosphere with a spherical lower boundary and with an idealized representation of thermodynamic processes as Newtonian relaxation of temperatures toward an axially and hemispherically symmetric radiative equilibrium state. The primitive equations are integrated with the spectral-transform method with T42 horizontal resolution and with 30 vertical levels — a resolution sufficient to represent the energy-containing baroclinic eddies. Model parameters are chosen such that features of the simulated climate, such as the mean meridional mass flux along isentropes, reproduce features of the Earth climate in an idealized fashion. The model will be described in detail in a forthcoming paper (Schneider and Held 2002). Included in Fig. 3 are the 5%, 50%, and 95% isolines of the cumulative distribution $\Pi_\phi^{\theta_s}(\theta)$ of surface potential temperatures. The 50% isoline, the median, approximates the mean surface potential temperature, and the 5% and 95% isolines can be taken as demarcating the surface layer.

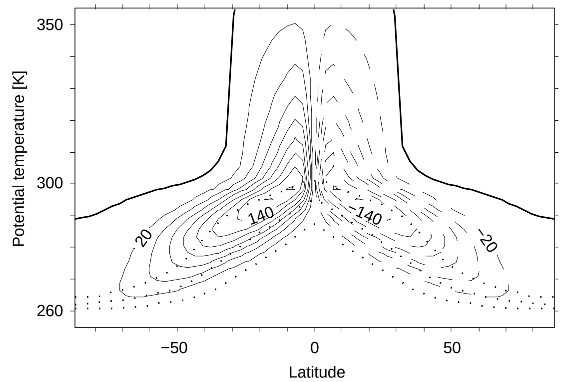

FIGURE 3: Mass flux streamfunction Ψ $[10^9\ \mathrm{kg\,s^{-1}}]$ in a simulation with an idealized GCM (solid lines: counterclockwise rotation; dashed lines: clockwise rotation). The dotted lines represent the 5%, 50%, and 95% isolines of the cumulative distribution $\Pi_\phi^{\theta_s}(\theta)$ of surface potential temperatures. The thick lines mark the mean position of the tropopause.

The streamfunction of the mean isentropic mass flux is characterized by an overturning cell in each hemisphere, with equatorward mass flux in the surface layer and poleward mass flux in the interior atmosphere. As discussed by Held and Schneider (1999), qualitative aspects of the mass flux streamfunction can be understood by assuming that eddies tend to homogenize quantities that are materially conserved in adiabatic and inviscid flows: In the interior atmosphere, downgradient mixing of potential vorticity leads to a southward eddy flux $\overline{\hat{v}\hat{P}}^*$ of interior potential vorticity, which, according to the balance condition (45), is associated with a poleward mass flux. And at the surface, downgradient mixing of potential temperature leads to a poleward eddy flux $\overline{\tilde{v}_s'\theta_s'}^s$ of potential temperature, which, according to the balance condition (45), is associated with an equatorward mass flux in the surface layer.

Figure 4 shows to what extent, in the simulation with the idealized GCM, the balance condition (45) is quantitatively accurate at the top of the surface layer. Dis-

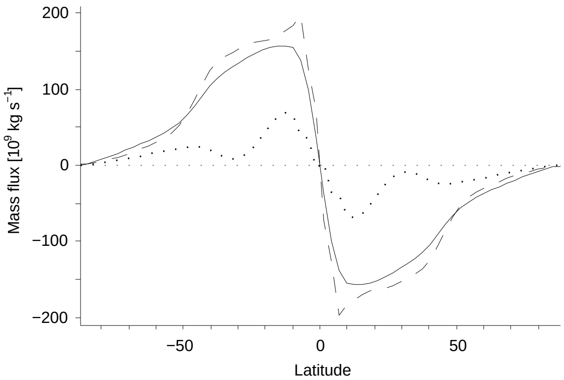

FIGURE 4: Mean meridional mass flux integrated over surface layer (up to the 95% isoline of the cumulative distribution $\Pi_\phi^{\theta_s}(\theta)$ of surface potential temperatures). The solid line represents the actual mass flux [left-hand side of Eq. (45)]; the dashed line the approximate mass flux due to eddy fluxes and friction [right-hand side of Eq. (45)]; and the dotted line the mass flux due to the eddy component and frictional component of the interior potential vorticity flux [integral on the right-hand side of Eq. (45)]. The mass fluxes are multiplied by the lengths $2\pi\cos(\phi)$ of latitude circles, so that they are comparable with the values of the streamfunction in Fig. 3.

played in Fig. 4 are terms in the balance condition (45) with the top of the surface layer as the upper limit $\theta_i(\phi)$ of the integration; the mass fluxes displayed thus are mass fluxes integrated over the surface layer. The top of the surface layer is taken to be the 95% isoline of the cumulative distribution $\Pi_\phi^{\theta_s}(\theta)$ of surface potential temperatures (the uppermost dotted line in Fig. 3). In midlatitudes, neglecting the diabatic component $\overline{J_Q^y}^*$ of the interior potential vorticity flux and approximating the mean interior potential vorticity $\overline{P}^*$ in the integration of the surface potential vorticity flux by a function f/ρ_θ^0 of only latitude can be seen to lead to errors of less than 20% in the approximate mean meridional mass flux (45). In the tropics (at latitudes $|\phi| < 30°$), the agreement between the actual mean meridional mass flux and the approximate mean meridional mass flux (45) also appears to be close, but this close agreement is coincidental: the errors incurred by neglecting the diabatic component $\overline{J_Q^y}^*$ of the interior potential vorticity flux and by approximating the mean interior potential vorticity in the integration of the surface potential vorticity flux by $\overline{P}^* \approx f/\rho_\theta^0$ are individually relatively large but cancel partially in their contribution to the mean meridional mass flux. In the extratropical surface layer, the eddy component $\overline{\hat{v}\hat{P}}^*$ partially balances the frictional component $\overline{J_F^y}^*$ of the interior potential vorticity flux, and the net mass flux due to these components of the interior potential vorticity flux [the integral on the right-hand side of Eq. (45)] is much smaller than the mass flux due to the eddy flux $\overline{\tilde{v}_s'\theta_s'}^s$ of surface potential temperature. The smallness of the net mass flux due to the eddy component $\overline{\hat{v}\hat{P}}^*$ and the frictional component $\overline{J_F^y}^*$ of the interior potential vorticity flux in the surface layer suggests that this mass flux can be neglected in a first approximation. [In the interior atmosphere, on the other hand, the mass flux due to the eddy flux $\overline{\hat{v}\hat{P}}^*$ of interior potential vorticity dominates the mean meridional mass flux along isentropes (cf. Held and Schneider 1999).]

Therefore, inasmuch as the approximate balance condition (45) reflects the dominant contributions to the mean meridional mass flux in the extratropics, developing a theory of the mean meridional mass flux along isentropes in the extratropics involves developing theories for the eddy fluxes of interior potential vorticity along isentropes and of potential temperature along the surface (Held and Schneider 1999). Since the problem of understanding the thermal stratification of the atmosphere is equivalent to the problem of understanding the distributions of potential temperature along the surface and of mass along isentropes, a theory of the mean meridional mass flux along isentropes will be equivalent to a theory of the thermal stratification. A theory of the extratropical thermal stratification that is based on the balance condition (45), paired with theories for the eddy fluxes of interior potential vorticity and surface potential temperature, will be proposed in a forthcoming paper (Schneider and Held 2002).

7. Summary

We have presented a formulation of potential vorticity dynamics that encompasses boundary effects. For arbitrary flows, the generalization of the potential vorticity concept to a sum of the conventional interior potential vorticity and a singular surface potential vorticity allows one to replace the inhomogeneous boundary condition for potential vorticity dynamics by a simpler homogeneous boundary condition. For the generalized potential vorticity, a conservation law holds that is similar to the well-known conservation law for the interior potential vorticity. The generalized potential vorticity reduces in the quasigeostrophic limit to Bretherton's (1966) generalized quasigeostrophic potential vorticity, which includes a surface potential vorticity that is proportional to surface potential temperature fluctuations. Not being limited to quasigeostrophic flows, however, the generalized potential vorticity concept can be used to describe flows such as mesoscale or planetary-scale flows, for which the quasigeostrophic approximation is inadequate.

The formal framework of generalized potential vorticity dynamics issues from field equations in which the potential vorticity and the potential vorticity flux appear as sources of flow fields in the same way in which an electric charge and an electric current appear as sources of fields in electrodynamics. The field equations make manifest that the potential vorticity and the potential vorticity flux can be interpreted as sources of absolute angular momentum and of energy of the flow along isentropes. The boundary sources — the surface potential vorticity and the surface potential vorticity flux — that must be included in the field equations if the usual inhomogeneous boundary condition for potential vorticity dynamics is replaced by a simpler homogeneous boundary condition were determined with techniques from electrodynamics. We derived functional forms of the surface potential vorticity and of its flux, pointed out ambiguities in these functional forms, and discussed the conservation law for the generalized potential vorticity.

In two examples, we demonstrated how the generalized potential vorticity and its conservation law can be used to analyze the dynamical role of boundaries in flows for which the quasigeostrophic approximation is inadequate.

First, we outlined a theory of how a wake with lee vortices can form in flows past a mountain that has no adjacent frictional boundary layer. Even in adiabatic and frictionless flows, generalized potential vorticity is not, in general, materially conserved but can be induced by baroclinicity at a boundary. In stratified flows past a sufficiently high mountain, generalized potential vorticity can be induced by baroclinicity on the leeward slope of the mountain. As illustrated in a simulation of a stratified Boussinesq flow, weak dissipative processes suffice to separate a baroclinically induced surface potential vorticity dipole from the leeward slope of a mountain and to advect it as an interior potential vorticity dipole along isentropes that intersect the surface. Thus a wake with a pair of counterrotating lee vortices can form by separation of a baroclinically induced generalized potential vorticity sheet from the surface of a mountain, even when the mountain has no adjacent frictional boundary layer from which friction-induced vorticity could be transferred into the wake in the interior of the flow.

Second, we derived a balance condition that relates the extratropical mean meridional mass flux along isentropes to eddy fluxes of interior potential vorticity and of surface potential temperature. Replacing the inhomogeneous boundary condition of fluctuating potential temperature by a homogeneous boundary condition of constant potential temperature formally renders the entropic domain of generalized potential vorticity dynamics time-independent. Thus the generalized potential vorticity concept allows the consideration of a mean potential vorticity budget throughout the entire entropic flow domain, including the surface layer of isentropes that sometimes intersect the surface. For statistically stationary and axisymmetric flows, the mean budget of generalized potential vorticity implies an approximate balance condition between the mean meridional mass flux along isentropes on the one hand and eddy fluxes of interior potential vorticity along isentropes and of potential temperature along the surface on the other hand. In the extratropical surface layer, the approximations that lead to this balance condition incurred errors of less than 20% in the mean meridional mass flux simulated with an idealized GCM. Since the problem of understanding the thermal stratification of the atmosphere is equivalent to the problem of understanding the distributions of potential temperature along the surface and of mass along isentropes, a theory of the extratropical thermal stratification can be based upon the balance condition between the mean meridional mass flux and the eddy fluxes, paired with theories for the eddy fluxes. Such a theory will be proposed in a forthcoming paper (Schneider and Held 2002).

These two examples illustrate how the generalized potential vorticity concept extends the conventional potential vorticity concept to encompass boundary effects in flows that need not be balanced. For balanced flows, the inversion principle known from conventional potential vorticity dynamics carries over to generalized potential vorticity dynamics: like the conventional potential vorticity combined with typically inhomogeneous boundary conditions, the generalized potential vorticity combined with simpler homogeneous boundary conditions contains all relevant information about flows that satisfy fairly general balance conditions (cf. Hoskins et al. 1985; McIntyre and Norton 2000). Therefore, as Bretherton's extension of the quasigeostrophic potential vorticity contains all relevant information about quasigeostrophic flows and their boundary conditions, the generalized potential vorticity contains all relevant information about more general balanced flows and their boundary conditions.

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APPENDIX

Notation and Symbols

Most symbols follow standard meteorological and mathematical conventions. Listed here are only those symbols that are used repeatedly in different sections of this paper.

∂_s	Partial derivative with respect to space or time coordinate s
D/Dt	Material derivative $D/Dt = \partial_t + \mathbf{u} \cdot \boldsymbol{\nabla}$ following the three-dimensional flow $\mathbf{u}$
$\overline{(\cdot)}$	Mean
$(\cdot)'$	Fluctuation $(\cdot)' = (\cdot) - \overline{(\cdot)}$ about mean $\overline{(\cdot)}$
$\overline{(\cdot)}^*$	Mean $\overline{(\rho_\theta \,\cdot)}/\bar{\rho}_\theta$ weighted by isentropic density ρ_θ
$\hat{(\cdot)}$	Fluctuation $\hat{(\cdot)} = (\cdot) - \overline{(\cdot)}^*$ about density-weighted mean $\overline{(\cdot)}^*$
c_p	Specific heat at constant pressure
E	Exner function $E = c_p(p/p_0)^\kappa$
f	Coriolis parameter $f = 2\|\boldsymbol{\Omega}\| \sin\phi$, where ϕ is latitude
g	Gravitational acceleration
$\mathbf{F}$	Frictional force per unit mass
h	Scale factor $h = (\partial_z \theta)^{-1}$ of isentropic coordinates
$\mathcal{H}(\cdot)$	Heaviside step function
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	Local Cartesian unit vectors (eastward, northward, upward)
$\mathbf{J}, \mathbf{J}_g$	Potential vorticity flux, generalized potential vorticity flux
$\mathbf{K}$	Surface potential vorticity flux
M	Montgomery streamfunction $M = c_pT + gz$
$\mathbf{n}$	Unit normal vector at the surface (directed upward)
p, p_0	Pressure, constant reference pressure
P, P_g	Potential vorticity, generalized potential vorticity
Q	Diabatic heating rate $Q = D\theta/Dt$
$\mathbf{r}$	Radius vector $\mathbf{r} = \mathbf{r}(\mathbf{x})$ from center of the planet to point $\mathbf{x}$ in the atmosphere
R	Gas constant
S	Surface potential vorticity
t	Time
T	Temperature
u, v, w	Velocity components (eastward, northward, upward)
$\mathbf{u}$	Three-dimensional velocity [$\mathbf{u} = (u, v, w)$ in local Cartesian coordinates]
$\mathbf{u_a}$	Three-dimensional absolute velocity $\mathbf{u_a} = \mathbf{u} + \boldsymbol{\Omega} \times \mathbf{r}$
$\mathbf{v}$	Horizontal velocity [$\mathbf{v} = (u, v, 0)$ in local Cartesian coordinates]
x, y, z	Local Cartesian coordinates (eastward, northward, upward)
$\mathbf{x}$	Three-dimensional position vector [$\mathbf{x} = (x, y, z)$ in local Cartesian coordinates]
$\delta(\cdot)$	Dirac delta-function
$\zeta_\sigma, \zeta_\theta$	Relative vorticity of horizontal flow $\mathbf{v}$ along surface $\left[\zeta_\sigma = \mathbf{n} \cdot (\boldsymbol{\nabla} \times \mathbf{v})\right]$ and along isentropes $\left[\zeta_\theta = \boldsymbol{\nabla}\theta \cdot (\boldsymbol{\nabla} \times \mathbf{v})/\|\boldsymbol{\nabla}\theta\|\right]$
θ, θ_0	Potential temperature $\theta = T(p_0/p)^\kappa$, constant reference potential temperature
θ_s^x, θ_s^y	Derivative $\partial_{x,y}\,\theta_s(x, y, t)$ of surface potential temperature $\theta_s(x, y, t)$ with respect to x, y
κ	Adiabatic exponent $\kappa = R/c_p$
μ	Normalization factor $\mu = \left(1 + \|\boldsymbol{\nabla} z_s\|^2\right)^{-1/2}$
ϕ	Latitude
Φ	Geopotential
$\pi_{\mathbf{x}}^{\psi}(\cdot)$	Probability density function of variable ψ at position $\mathbf{x}$
$\Pi_{\mathbf{x}}^{\psi}(\cdot)$	Cumulative distribution function of variable ψ at position $\mathbf{x}$
ρ	Density
ρ_θ	Isentropic density $\rho_\theta = g^{-1}\partial_\theta p$ [density in (x, y, θ)-space]
$\boldsymbol{\omega}_\mathbf{r}$	Relative vorticity $\boldsymbol{\omega}_\mathbf{r} = \boldsymbol{\nabla} \times \mathbf{u}$
$\boldsymbol{\omega}_\mathbf{a}$	Absolute vorticity $\boldsymbol{\omega}_\mathbf{a} = \boldsymbol{\omega}_\mathbf{r} + 2\boldsymbol{\Omega}$
$\boldsymbol{\Omega}$	Angular velocity of planetary rotation

REFERENCES

Andrews, D. G., J. R. Holton, and C. B. Leovy, 1987: *Middle Atmosphere Dynamics*. International Geophysics Series, Vol. 40, Academic Press, 489 pp.

Arfken, G., 1985: *Mathematical Methods for Physicists*. 3d ed. Academic Press, 985 pp.

Batchelor, G. K., 1967: *An Introduction to Fluid Dynamics*. Cambridge University Press, 615 pp.

Bretherton, C. S., and C. Schär, 1993: Flux of potential vorticity substance: A simple derivation and a uniqueness property. *J. Atmos. Sci.*, **50**, 1834–1836.

Bretherton, F. P., 1966: Critical layer instability in baroclinic flows. *Quart. J. Roy. Meteor. Soc.*, **92**, 325–334.

Gallimore, R. G., and D. R. Johnson, 1981: The forcing of the meridional circulation of the isentropic zonally averaged circumpolar vortex. *J. Atmos. Sci.*, **38**, 583–599.

Hallberg, R., and P. B. Rhines, 2000: Boundary sources of potential vorticity in geophysical circulations. *Developments in Geophysical Turbulence*, R. M. Kerr and Y. Kimura, Eds., Kluwer.

Haynes, P. H., and M. E. McIntyre, 1987: On the evolution of vorticity and potential vorticity in the presence of diabatic heating and frictional or other forces. *J. Atmos. Sci.*, **44**, 828–841.

——, and ——, 1990: On the conservation and impermeability theorems for potential vorticity. *J. Atmos. Sci.*, **47**, 2021–2031.

Held, I. M., and T. Schneider, 1999: The surface branch of the zonally averaged mass transport circulation in the troposphere. *J. Atmos. Sci.*, **56**, 1688–1697.

Hoskins, B. J., M. E. McIntyre, and A. W. Robertson, 1985: On the use and significance of isentropic potential vorticity maps. *Quart. J. Roy. Meteor. Soc.*, **111**, 877–946.

Jackson, J. D., 1975: *Classical Electrodynamics*. 2d ed. Wiley, 848 pp.

Marshall, J. C., and A. J. G. Nurser, 1992: Fluid dynamics of oceanic thermocline ventilation. *J. Phys. Oceanogr.*, **22**, 583–595.

——, D. Jamous, and J. Nilsson, 2001: Entry, flux, and exit of potential vorticity in ocean circulation. *J. Phys. Oceanogr.*, **31**, 777–789.

McDougall, T. J., 1988: Neutral-surface potential vorticity. *Prog. Oceanogr.*, **20**, 185–221.

McIntyre, M. E., and W. A. Norton, 2000: Potential vorticity inversion on a hemisphere. *J. Atmos. Sci.*, **57**, 1214–1235.

Monin, A. S., and A. M. Yaglom, 1971: *Statistical Fluid Mechanics: Mechanics of Turbulence*. Vol. 1, MIT Press, 769 pp. Updated, augmented, and revised translation of *Statisicheskaya gidromekhanika — Mekhanika Turbulentnosti* (Nauka Press, Moscow, 1965).

Morse, P. M., and H. Feshbach, 1953: *Methods of Theoretical Physics*. International Series in Pure and Applied Physics, McGraw-Hill, 1978 pp. (2 vols.).

Papoulis, A., 1991: *Probability, Random Variables, and Stochastic Processes*. 3d ed. McGraw Hill, 666 pp.

Rhines, P. B., 1979: Geostrophic turbulence. *Ann. Rev. Fluid Mech.*, **11**, 401–441.

——, and W. R. Young, 1982: Homogenization of potential vorticity in planetary gyres. *J. Fluid. Mech.*, **122**, 347–367.

Rotunno, R., V. Grubišić, and P. K. Smolarkiewicz, 1999: Vorticity and potential vorticity in mountain wakes. *J. Atmos. Sci.*, **56**, 2796–2810.

Salmon, R., 1998: *Lectures on Geophysical Fluid Dynamics*. Oxford University Press, 378 pp.

Schär, C., 1993: A generalization of Bernoulli's theorem. *J. Atmos. Sci.*, **50**, 1437–1443.

——, and D. R. Durran, 1997: Vortex formation and vortex shedding in continuously stratified flows past isolated topography. *J. Atmos. Sci.*, **54**, 534–554.

Schneider, T., and I. M. Held, 2002: The tropopause and the thermal stratification in the extratropics of a dry atmosphere. In preparation.

Smith, R. B., 1988: Linear theory of stratified flow past an isolated mountain in isosteric coordinates. *J. Atmos. Sci.*, **45**, 3889–3896.

——, 1989: Comment on "Low Froude number flow past three-dimensional obstacles. Part I: Baroclinically generated lee vortices." *J. Atmos. Sci.*, **46**, 3611–3613.

Smolarkiewicz, P. K., and R. Rotunno, 1989: Low Froude number flow past three-dimensional obstacles. Part I: Baroclinically generated lee vortices. *J. Atmos. Sci.*, **46**, 1154–1164.

Tung, K. K., 1986: Nongeostrophic theory of zonally averaged circulation. Part I: Formulation. *J. Atmos. Sci.*, **43**, 2600–2618.